Hyperbolic Spaces and Ptolemy Möbius Structures

Dissertation

zur

Erlangung der naturwissenschaftlichen Doktorwürde
(Dr.sc.nat)

vorgelegt der

Mathematisch-naturwissenschaftlichen Fakultät

der

Universität Zürich

von

Renlong Miao

von

China

Promotionskomitee

Prof. Dr. Viktor Schroeder(Vorsitz)
Prof. Dr. Sergei Buyalo

Zürich 2013

Abstract

We mainly study the relationship between the CAT(κ) spaces and PT$_\kappa$ spaces. It's easy to show that the CAT(κ) spaces are PT$_\kappa$ spaces while the vice verse is not true. In [FLS] there gives an counterexample for a PT$_0$ geodesic space which is not unique geodesic. We show the under the κ-busemann convexity PT$_\kappa$ spaces are CAT(κ) spaces.

We generalize the PT$_\kappa$ spaces to asymptotically PT$_\kappa$ spaces. Later we show some very nice properties for asymptotically PT$_\kappa$ spaces for $\kappa < 0$. Such as Gromov hyperbolicity and boundary continuality. We also prove that let $(Z, \mathcal{M})$ be a complete and ptolemaic Möbius space. Then there exists an asymptotically PT$_{-1}$ space X such that $\partial_\infty X$ with its canonical Möbius structure is Möbius equivalent to $(Z, \mathcal{M})$. As a very important corollary, we prove that every visual Gromov hyperbolic space is roughly similar to some asymptotically PT$_{-1}$ space.

We also prove a more general flat strip theorem for PT$_0$ spaces without the constraint of Busemann convexity. In case of p-uniformly convex spaces, we show a similar Jung's theorem and fixed point theorems.

Zusammenfassung

Wir studieren die Beziehung zwischen CAT(κ) und PT$_\kappa$-Räumen. Es ist einfach zu zeigen, dass CAT(κ)-Räume auch PT$_\kappa$ sind, während die Umkehrung nicht gilt. In [FLS] wird ein PT$_0$-Raum konstruiert, der nicht eindeutig geodätisch ist. Wir zeigen, dass ein PT$_\kappa$-Raum der zusätzlich κ-Busemann konvex ist, ein CAT(κ)-Raum ist.

Wir verallgemeinern den Begriff des PT$_\kappa$-Raumes und definieren *asymptotische* PT$_\kappa$-Räume. Wir zeigen, dass diese (für $\kappa < 0$) Gromov hyperbolisch und randstetig sind. Weiter beweisen wir, dass es zu einem kompakten ptolemäischen Möbius Raum $(Z, \mathcal{M})$ einen asymptotischen PT$_{-1}$-Raum X gibt, so dass $\partial_\infty X$ mit seiner kanonischen Möbiusstruktur Möbius equivalent zu $(Z, \mathcal{M})$ ist. Als Korollar erhalten wir, das jeder sichtbare Gromov hyperbolische Raum grob ähnlich zu einem asymptotisch PT$_{-1}$-Raum ist.

Wir beweisen einen *Flachen Streifen Satz* für geodätische ptolomäische Räume sowie ein Jung Theorem und Fixpunktsätze für p-uniform konvexe Räume.

Acknowledgements

First and foremost I want to thank my advisor, Prof. Viktor Schroeder. It has been my honor to be his Ph.D student. He has taught me, both consciously and unconsciously, how a good math is done. I appreciate all his contributions of time, ideas, discussions and funding to make my Ph.D experience productive and smoothly. I am thankful for the excellent example he has made not only as a successful mathematician but also the responsibility for a man in the family.

My time at Zurich was made enjoyable in large part due to the friends and groups that became a part of my life. I am grateful for the time spent with officemates and colleagues. My time at Zurich was also enriched by the soccer events with all the friends.

Lastly, I would like to thank my family for all their love and encouragement. For my parents who raised me with a love of math and supported me in all my pursuits. For my sister who listened to me all the time.

Thank you all!

Contents

Chapter 1

Introduction and Outline

The starting point in the theory of Gromov hyperbolic spaces is an ingenious observation by Gromov that quadruples of points in the standard hyperbolic space $\mathbb{H}^n$ satisfy a condition, the so-called *δ-inequality*, which takes into account only properties of the metric. Consequently, this condition can be taken as a *definition* of hyperbolicity of arbitrary metric spaces. What is surprising is that this simple condition, while making absolutely no requirements on the space on any bounded scale, imposes strong conditions on the space on large scales that make it behave very similarly to negatively curved manifolds. Gromov hyperbolic spaces have been extensively studied over the last two decades, mostly with regard to geometric group theory. Some standard references include [G1], [BH], [BS1]. There is a deep and well studied relation between the geometry of the classical hyperbolic space and the Möbius geometry of its boundary at infinity. This relation can be generalized in a nice way to CAT(-1) spaces.

Let X be a CAT(-1) space with boundary $Z = \partial_\infty X$. For every basepoint $o \in X$ one can define the Bourdon metric $\rho_o(x,y) = e^{-(x|y)_o}$ on Z, where $(\,|\,)_o$ is the Gromov product with respect to o, compare [BS1]. For different basepoints $o, o' \in X$ the metrics $\rho_o, \rho_{o'}$ are Möbius equivalent and thus define a Möbius structure on Z. By [FS3] this Möbius structure is ptolemaic.

On the other hand, examples show that not every ptolemy Möbius structure arises as boundary of a CAT(-1) space. In this paper we enlarge the class of CAT(-1) spaces in a way that this larger class corresponds exactly to the spaces which have a ptolemaic Möbius structure at infinity.

Definition 1. *A metric space is called* asymptotically PT$_{-1}$*, if there exists some $\delta > 0$ such that for all quadruples $x_1, x_2, x_3, x_4 \in X$ we have*

$$e^{\frac{1}{2}(\rho_{1,3}+\rho_{2,4})} \;\leq\; e^{\frac{1}{2}(\rho_{1,2}+\rho_{3,4})} \;+\; e^{\frac{1}{2}(\rho_{1,4}+\rho_{2,3})} \;+\; \delta\, e^{\frac{1}{2}\rho},$$

where $\rho_{i,j} = d(x_i, x_j)$ and $\rho = \max_{i,j} \rho_{i,j}$.

It turns out that CAT(-1) are asymptotically PT$_{-1}$ and that the relation between these spaces and the Möbius geometry of their boundaries can be expressed in the following two results:

Theorem 2. *Let X be asymptotically* PT$_{-1}$*, then X is a boundary continuous Gromov hyperbolic space. For every base point $o \in X$, $\rho_o(x,y) = e^{-(x|y)_o}$ defines a metric on $\partial_\infty X$. For different base points these metrics are Möbius equivalent* *and thus define a canonical Möbius structure $\mathcal{M}$ on $\partial_\infty X$. The Möbius structure $\mathcal{M}$ is complete and ptolemaic.*

Theorem 3. *Let $(Z, \mathcal{M})$ be a complete and ptolemaic Möbius space. Then there exists an asymptotically* PT_{-1} *space X such that $\partial_\infty X$ with its canonical Möbius structure is Möbius equivalent to $(Z, \mathcal{M})$.*

In the proof we use a hyperbolic cone construction due to Bonk and Schramm [BS]. which associates to a metric space (Z, d) a cone $(\mathrm{Con}(Z), \rho)$. We show in Proposition 134 that if (Z, d) is ptolemaic, then the cone is asymptotically PT_{-1}. This method can also be used obtain a characterization of visual Gromov hyperbolic spaces in the spirit of the [BS]. Recall that two metric spaces X and Y are roughly similar, if there are constants $K, \lambda > 0$ and a map $f : X \to Y$ such that for all $x, y \in X$

$$|\lambda d_X(x, y) \; - \; d_Y(f(x), f(y))| \; \leq K$$

and in additional $\sup_{y \in Y} d_Y(y, f(X)) \leq K$.

A theorem of Bonk and Schramm states that a visual Gromov hyperbolic space with doubling boundary is roughly similar to a convex subset of the real hyperbolic space H^n for some integer n.

We have a version without conditions on the boundary:

Theorem 4. *Every visual Gromov hyperbolic space is roughly similar to some asymptotically* PT_{-1} *space.*

Later we have done some investigations on the convexity of $\mathrm{CAT}(\kappa)$ spaces, hence with a more general extension to the PT_κ spaces. Moreover we have given another characterization of $\mathrm{CAT}(\kappa)$ spaces with the convexities we introduced. For PT_0 space, we show a more general flat strip theorem without the constraint of Busemann convexity.

Theorem 5. *Let X be a geodesic* PT_0 *space which is homeomorphic to* $\mathbb{R} \times [0, 1]$*, such that the boundary curves are parallel geodesic lines, then X is isometric to a flat strip* $\mathbb{R} \times [0, a] \subset \mathbb{R}^2$ *with its euclidean metric.*

For the last part of the thesis we explain some extension theorems for gromov hyperbolic spaces and compared the two different hyperbolic cone constructions. We introduce the work which is mainly done by Viktor Schroeder and S. Buyalo in [BS1].

The whole structure of this thesis is by the following: after introducing some preliminary notation and remarks, we recall in Ch. 3 the basics of Gromov hyperbolic metric spaces. In Ch. 4 we introduce the different morphisms between Gromov hyperbolic spaces. In Ch. 5 we discuss the Boundary at Infinity of such a space. We also discuss the notion of asymptotic curvature of a metric space and give an example of a phenomenon that so far has not appeared in the literature. In Ch. 6 is the heart of this thesis and here we introduce the definition of asymptotic PT_{-1} space and develop some properties of it. In Ch. 10 we recall and generalize a construction of Buyalo and Schroeder [BS1] to produce a Gromov hyperbolic space with prescribed Boundary at Infinity and also compare it with the hyperbolic cones which is proposed in [BS].

Chapter 2

Preliminaries

Here we gather some well known terminology and define some general terms that are so ubiquitous throughout the text that we find it appropriate to define them right away.

A *metric space* is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfies

1. $d(x, y) = 0 \Leftrightarrow x = y$,
2. $d(x, y) = d(y, x)\ \forall x, y \in X$,
3. $d(x, z) \leq d(x, y) + d(y, z)\ \forall x, y, z \in X$.

A metric space is called *proper* if closed and bounded sets are compact.

Remark 6. *We often find it convenient to use the notation $|xy|$, or even just xy for $d(x, y)$. It will be clear from the context that this refers to the distance between the points.*

The *Hausdorff distance* of two subsets $A, B \subset X$ of a metric space, denoted $d_H(A, B)$, is defined by

$$d_H(A, B) := \inf\{\epsilon > 0 \mid A \subset B_\epsilon \text{ and } B \subset A_\epsilon\},$$

where A_ϵ, B_ϵ are the *closed* ϵ-neighborhoods of A and B respectively, i.e. $A_\epsilon := \{x \in X \,|\mathrm{dist}(x, A) \leq \epsilon\}$.

If $A, B \subset X$ are subsets of a metric space, we say that A is a C*-net* in B, or A is C*-cobounded* in B, if $d_H(A, B) \leq C$. If we just say that A is a net in B, we mean that there exists a C such that A is a C-net in B.

Example 7. *$\mathbb{Z}$ is a $1/2$-net in $\mathbb{R}$.*

Definition 8. *A triple of real numbers $\{a, b, c\}$ is called a (additive) δ-*triple *if the two* smaller *numbers differ by at most δ. For example, if $a \leq b \leq c$, then we require $|a - b| \leq \delta$.*

The multiplicative version is

Definition 9. *A triple of real numbers $\{a, b, c\}$ is called a (multiplicative) K-*triple *if the two* larger *numbers have ratio at most K, i.e. $c/b \leq K$ if we again assume $a \leq b \leq c$.*

A metric space $(X, |\cdot|)$ that satisfies $|xz| \leq \max\{|xy|, |yz|\}$ for all $x, y, z \in X$ is commonly called an *ultrametric space*. Generalizing this inequality leads to *quasimetric spaces*.

Definition 10. *A K-*quasimetric space *is a set Z together with a map $\rho : Z \times Z \to [0, \infty]$ such that*

1. *$\rho(z, y) \geq 0$ $\forall z, y \in Z$, with equality iff $y = z$,*
2. *$\rho(z, y) = \rho(y, z)$ $\forall z, y \in Z$,*
3. *$\rho(z, w) \leq K \max\{\rho(z, y), \rho(y, w)\}$ $\forall w, y, z \in Z$,*
4. *There is at most one $z \in Z$ such that $\rho(z, y) = \infty$ for all $y \in Z \setminus \{z\}$.*

If no point z as in 4 exists, Z is said to be non-extended*, while it is* extended *if there is such a z and this z is then called the* infinitely remote point*. By convention, a one-point space $Z = \{z\}$ is never extended.*

Property 3 above is equivalent to $\{\rho(x, y), \rho(x, z), \rho(y, z)\}$ being a multiplicative K-triple for any $x, y, z \in Z$.

A quasimetric ρ on a space Z induces a topology by declaring a set $A \subset Z$ to be open if for every $a \in A \setminus \{\infty\}$ there exists $r > 0$ such that $B_r^\rho(a) \subset A$, and if $\infty \in A$, then there exists $y \in Z$ and $r > 0$ such that $B_r(y)^c \subset A$. This topology is metrizable and in particular first-countable and Hausdorff. This follows from the fact that if (Z, ρ) is K-quasimetric, then (Z, ρ^s) is K^s-quasimetric (and the two topologies are clearly equivalent), and a result of Frink's ([F]) whereby a K-quasimetric with $1 \leq K \leq 2$ is bilipschitz equivalent to a metric (extended if ρ is extended).

Here and in the future we always denote $B_r^\rho(x) := \{z \in Z | \rho(z, x) < r\}$ the *quantitatively* open balls, while $\overline{B}_r^\rho(x) := \{z \in Z | \rho(z, x) \leq r\}$ are the *quantitatively* closed balls. Note, though, that in contrast to the metric setting quantitatively open (closed) balls need not be topologically open (closed). For example, consider $Z := [0, 1] \cup \{p\}$ with the 2-quasimetric ρ defined as the Euclidean distance for points on $[0, 1]$, $\rho(p, t) := 1$ for $t \in [0, 1/2]$ and $\rho(p, t) = 2$ for $t \in (1/2, 1]$. Then $B_{3/2}(p) = [0, 1/2] \cup \{p\}$ is not open.

Definition 11 (Completeness of a quasimetric)**.** *A quasimetric space (Z, ρ) is called* complete *if every Cauchy sequence in Z converges and if ρ is extended in case it is unbounded.*

We give some examples of quasimetric spaces.

Example 12.

1. *Every metric space is a 2-quasimetric space.*
2. *Every ultrametric space is a 1-quasimetric space.*
3. *The circle S^1 and $\mathbb{R} \cup \{\infty\}$ are complete quasimetric spaces, but $\mathbb{R}$ is not complete.*
4. *For the most important example, the boundary at infinity of a Gromov hyperbolic space, we refer to Section 5.2.1. They will turn out to be complete quasimetric spaces.*

Chapter 3

Gromov Hyperbolic Geodesic Spaces

3.1 Geodesic Spaces and Slim Triangles

A map $f : X \to Y$ between metric spaces is said to be *isometric* if it preserves their distances, i.e. $|f(x)f(x')| = |xx'|$ for each $x, x' \in X$. Clearly, every isometric map is injective. If f is in addition surjective, it is called an *isometry*.

Definition 13. *A* geodesic segment *in a metric space* $(X, |\cdot|)$ *is an isometric map* $\gamma : [a, b] \to X$*, where* $[a, b]$ *is a compact interval in* $\mathbb{R}$.

A geodesic ray *is an isometric map* $\gamma : [a, \infty) \to X$*, while an isometric map* $\gamma : (-\infty, \infty) \to X$ *is called a* bi-infinite ray.

Remark 14. *We frequently abbreviate and just speak of "a geodesic* $\gamma \ldots$*". It will always be clear from the context whether we mean a finite segment, a ray or a bi-infinite ray.*

Definition 15. *A metric space* X *is called* geodesic *if there exists a geodesic segment between any two of its points, i.e.* $\forall x, y \in X \; \exists a$ *geodesic* $\gamma : [a, b] \to X$ *with* $\gamma(a) = x, \gamma(b) = y$.

A space is uniquely geodesic *if for every choice of* x, y *there is a unique such geodesic* γ.

Remark 16. *We often abuse notation and write* xy *for a geodesic from* x *to* y*. If the space is not uniquely geodesic,* xy *stands for one arbitrary, but fixed, geodesic segment from* x *to* y.

A *triangle* in a metric space is the union $\gamma_0 \cup \gamma_1 \cup \gamma_2$ of three geodesics $\gamma_i, i = 0, 1, 2$, such that the endpoint of γ_i coincides with the starting point of $\gamma_{(i+1) \bmod 3}$. We also write xyz for a triangle with vertices x, y and z, cf. Rem. 16. To avoid technical difficulties of degenerate cases, we assume that the domains of all three geodesics have interior, i.e. the geodesics actually do form a triangle and not merely a line segment or a point.

A basic fact about the hyperbolic plane $\mathbb{H}^2$ is that any triangle is δ*-slim* in the sense that there is a constant $\delta = \delta(\mathbb{H}^2)$ such that for any triangle $\gamma_1 \cup \gamma_2 \cup \gamma_3$, γ_1 is contained in the δ-neighborhood of $\gamma_2 \cup \gamma_3$.

This leads us to the following geometric definition of Gromov hyperbolicity, commonly attributed to Rips.

Definition 17. *A geodesic metric space is called* δ-hyperbolic *if for any triangle* $\gamma_1 \cup \gamma_2 \cup \gamma_3$*, one has that* γ_1 *is contained in the* δ*-neighborhood of* $\gamma_2 \cup \gamma_3$*.*

A geodesic metric space is called Gromov hyperbolic *if it is* δ*-hyperbolic for some* $\delta \geq 0$*.*

This definition has the disadvantage that, while very intuitive, it is not at all obvious how to extend it to spaces which are not necessarily geodesic. For this reason we choose another, but equivalent, approach.

Definition 18. *Suppose* xyz *is a triangle in an arbitrary metric space* X*. It is elementary to show that there exists a unique triple of points* $u \in xy, v \in yz, w \in zx$ *such that* $|xu| = |xw|$, $|yu| = |yv|$, $|zv| = |zw|$*. The points* u, v, w *are called the* equiradial points *of the triangle* xyz*.*

In fact, the points u, v, w *are determined by the quantities*

$$\begin{aligned}|xu| = |xw| &= \frac{1}{2}(|xy| + |xz| - |zy|),\\ |yu| = |yv| &= \frac{1}{2}(|yx| + |yz| - |xz|),\\ |zv| = |zw| &= \frac{1}{2}(|zy| + |zx| - |xy|).\end{aligned}$$

We make the following

Definition 19. *Let* X *a metric space,* $x, y, o \in X$*. The non-negative real number*

$$(x|y)_o := \frac{1}{2}(|ox| + |oy| - |xy|)$$

is called the Gromov product *of* x *and* y *w.r.t. the base point* o*.*

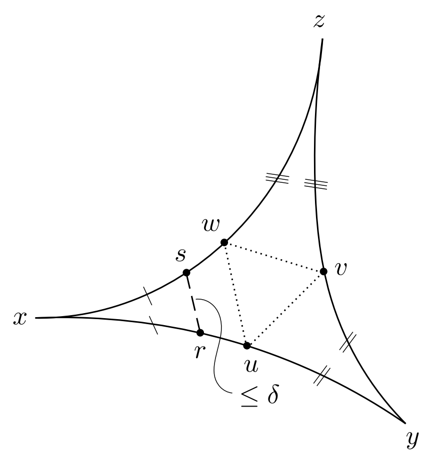


Figure 3.1: Equiradial points and thin triangles.

We have an elementary

Proposition 20. *Let X be a geodesic metric space that has the following property for some $\delta \geq 0$, cf. Fig. 3.1. Whenever xyz is a triangle in X and $r \in xy$, $s \in xz$ are points with $|xr| = |xs| \leq (y|z)_x$, then*

$$|rs| \leq \delta. \tag{A}$$

Then X is δ-hyperbolic in the sense of Def. 17. Conversely, any geodesic metric space that is δ_0-hyperbolic in the sense of Def. 17 satisfies condition (A) *with $\delta = 10\delta_0$. In particular, a geodesic metric space is Gromov hyperbolic if and only if it satisfies* (A) *for some $\delta \geq 0$.*

Proof. That a space which satisfies (A) for some δ is δ-hyperbolic in the sense of Def. 17 is obvious.

For the converse direction, note that the triangle inequality implies that the equiradial points u, v, w of xyz have pairwise distance no larger than $4\delta_0$. Let now $r \in xy, s \in xz$ with $|rx| = |sx| \leq (y|z)_x$ and consider the triangle xuw. Since $|uw| \leq 4\delta_0$, it follows from the triangle inequality that any $r \in xu$ with $|ru| \geq 5\delta_0$ must be δ_0-close to a point in wx and hence r must be $2\delta_0$-close to s. If $|ru| \leq 5\delta_0$ (and thus also $|ws| \leq 5\delta_0$), r may be δ_0-close to a point on uw. But then $|rs| \leq \delta_0 + 4\delta_0 + 5\delta_0 = 10\delta_0$. Hence X satisfies condition (A) with constant 10δ. □

We give a few examples.

Example 21.
1. *The standard hyperbolic spaces $\mathbb{H}^n$ of constant sectional curvature -1 are δ-hyperbolic for $\delta_{\mathbb{H}} = 2\ln\tau$, where τ is the solution of $t^2 = t + 1$.*
2. *Every* $\mathrm{CAT}(-1)$*-space is $\delta_{\mathbb{H}}$ hyperbolic.*
3. *A metric tree is* 0*-hyperbolic. Conversely, every* 0*-hyperbolic geodesic metric space is a metric tree.*

We end this section with one of the most fundamental results about geodesic hyperbolic spaces; the stability of geodesics or, rather, of quasigeodesics. The analogous fact for classical hyperbolic space was originally proved by Morse [M1] [M2]. In some sense, this theorem is really what makes the geometry of Gromov hyperbolic spaces accessible and allows to generalize many properties from classical hyperbolic space. For a proof, see for example [BH] Thm. III.H.1.7.

Theorem 22 (Stability of geodesics)**.** *Let X be a geodesic δ-hyperbolic metric space and $\gamma : [0, a] \to X$ a (c, d)-quasigeodesic. There exists a constant $H = H(\delta, c, d)$ such that if η is any geodesic from $\gamma(0)$ to $\gamma(a)$, then* $\mathrm{im}(\gamma)$ *and* $\mathrm{im}(\eta)$ *are H-close in Hausdorff distance.* □

Note that the constant H does *not* depend on a.

3.2 Hyperbolicity in General Metric Spaces

Property (A) from Prop. 20 leads to a definition of Gromov hyperbolicity in general metric spaces. The crucial point is the following δ-inequality, originally due to Gromov [G1].

Proposition 23 ([BS1] Prop. 2.1.2, 2.1.3)**.** *If a geodesic space X is δ-hyperbolic in the sense of property* (A)*, then*

$$(x|y)_o \geq \min\{(x|z)_o, (y|z)_o\} - \delta \tag{3.1}$$

for any $o, x, y, z \in X$.

Conversely, if a geodesic space X satisfies the δ-inequality (3.1) *for every $o, x, y, z \in X$, then it satisfies* (A) *with 4δ.* □

This allows us to give meaning to hyperbolicity in general metric spaces.

Definition 24 ([G1] 1.1)**.** *A metric space X is called* Gromov hyperbolic *if there is a δ such that every quadruple $o, x, y, z \in X$ of points in X satisfy the δ-inequality* (3.1)*.*

An equivalent formulation is as follows (recall Def. 8 for the notion of a δ-triple).

Proposition 25. *A metric space X is Gromov hyperbolic if and only if there is a δ such that for every quadruple of points in X, $o, x, y, z \in X$, the triple $\{(x|y)_o, (x|z)_o, (y|z)_o\}$ is a δ-triple.*

This is furthermore equivalent to the triple

$$\{-|xz| - |yo|, -|xo| - |yz|, -|xy| - |zo|\}$$

being a 2δ-triple for all $x, y, z, o \in X$.

Proof. Write out the definitions. □

Remark 26. *The triple $\{|xz| + |yo|, |xo| + |yz|, |xy| + |zo|\}$ is called the* cross-difference triple *of the quadruple (x, y, z, o).*

Chapter 4

Morphisms Between Gromov Hyperbolic Spaces

Here we introduce the morphisms between hyperbolic spaces and investigate their properties. These are roughly isometric maps, quasimöbius maps and quasisymmetric maps.

4.1 PQ-isometric maps

Definition 27. *A map $F : X \to Y$ between metric spaces X and Y is called* C-roughly isometric *if $|F(x)F(y)| \doteq_C |xy|$ $\forall x, y \in X$. Here $|F(x)F(y)| \doteq_C |xy|$ means $||F(x)F(y)| - |xy|| \leq C$.*

It is called roughly isometric *if it is C-roughly isometric for some C. It is called a* rough isometry *if there exists a D such that $F(X)$ is a D-net in Y.*

Definition 28. *A metric space X is called* C-roughly geodesic *if there exists for any $x, y \in X$ a* C-rough geodesic *joining x and y, where a C-rough geodesic is a C-roughly isometric map from an interval $I \subset \mathbb{R}$ into X.*

X is called roughly geodesic *if it is C-roughly geodesic for some C.*

To define an adequate setting for the definition of quasi- and power quasi-isometries, let us first introduce the notion of *cross-difference* of a quadruple of points.

Definition 29. *Let $Q = (x, y, z, w) \in X^4$ be an ordered quadruple in a metric space X. Define the* cross-difference *of Q, $cd(Q)$, as*

$$cd(Q) := (x|y)_o + (z|w)_o - (x|z)_o - (y|w)_o = \frac{1}{2}(|xz| + |yw| - |xy| - |zw|).$$

We will usually abuse notation and write $Q \subset X$ instead of $Q \in X^4$ for a quadruple in X, even though it is an *ordered* quadruple.

Definition 30. *Consider a map $F : X \to Y$ between metric spaces X and Y.*

F is called (c, d)-quasi-isometric *if*

$$\frac{1}{c}|xy| - d \leq |F(x)F(y)| \leq c|xy| + d \ \forall x, y \in X.$$

F is called (c,d)-power quasi-isometric *((c,d)-P-QI for short) if for all $cd(Q) \geq 0$,*

$$\frac{1}{c} cd(Q) - d \leq cd(F(Q)) \leq c \cdot cd(Q) + d \;\forall\; Q \in X^4.$$

F is called quasi-isometric *(*P-QI*) if it is (c,d)-quasi-isometric ((c,d)-P-QI) for some c, d.*

F is called a quasi-isometry *(*PQ-isometry*) if there exists a D such that $F(X)$ is a D-net in Y.*

Note that every P-QI map is also quasi-isometric (with the same constants). This follows from $cd(x,x,y,y) = |xy|$. Note also that every rough isometry, quasi-isometry and PQ-isometry has a *rough inverse* which is also a rough, quasi- or PQ-isometry respectively.

In the classical literature on hyperbolic spaces such as [G1], [BH], [BS1], only quasi-isometric maps are considered. The notion of P-QI maps was introduced in [BS1]. The problem with quasi-isometric maps is that in general they do not preserve hyperbolicity, see example 31 2. below. This is why Buyalo-Schroeder introduced P-QI maps, which, by their simultaneous control of distances between *four* points instead of only two, are the appropriate class of morphisms between Gromov hyperbolic spaces as we will see below. A striking result of Buyalo-Schroeder (Thm. 38 below) says that a quasi-isometric map between *geodesic* hyperbolic spaces is in fact P-QI. Since in the classical literature, which was concerned mainly with applications to geometric group theory, only geodesic spaces were considered, there was no need for the more general concept of a P-QI map.

Example 31. *1. $F : \{10^n \,|\, n \in \mathbb{N}\} \to \mathbb{R}$, $F(n) := (-1)^n 10^n$ is quasi-isometric, but not P-QI. Both the domain and image are hyperbolic.*

2. If $F : \{(x,y) \,|\, y = |x|\} \to \mathbb{R}$ is the projection onto the x-axis, then F is quasi-isometric (even bilipschitz). The domain is not hyperbolic, as is easily seen. The image, however, is. F can thus not be P-QI (cf. Thm. 38 below). This example is attributed to Väisälä.

3. Consider the following space X, built from a basic building block T, a six-point space that is a subset of a tree, as in Fig. 4.1 (only the points A, B, C, D, E, F belong to X, the edges are for illustration only). X is obtained by taking a series of T_i, where within T_i we have the distances:

$$|A_i E_i| = |B_i E_i| = |D_i F_i| = |C_i F_i| := 10^i, \;\; |E_i F_i| := i.$$

Furthermore, define $|C_{i-1} A_i| := 10^{10^i}$, and finally let all other distances in X be defined as the length of the shortest path between the two vertices. Then of course X is hyperbolic, 0-hyperbolic in fact.

Consider now the map $F : X \to X$ that switches B_i and D_i, but leaves all other points of X fixed.

This F actually has a property that places it somewhere in between ordinary quasi-isometric maps and bona-fide P-QI maps, namely in contrast *to general quasi-isometric maps F* does *preserve the Gromov product in the sense that (the constants $1/2$ and 2 are not optimal)*

$$\frac{1}{2}(x|y)_o \leq (F(x)|F(y))_{F(o)} \leq 2(x|y)_o.$$

This can be easily verified because in trees the Gromov product $(x|y)_o$ is the length of the geodesic segment from o to the point where the geodesics ox and oy branch.

However, F is not *P-QI, because $cd(A_i, B_i, C_i, D_i) = 0$ while*

$$cd(A'_i, B'_i, C'_i, D'_i) = cd(A_i, D_i, C_i, B_i) = |E_iF_i| = i \to \infty.$$

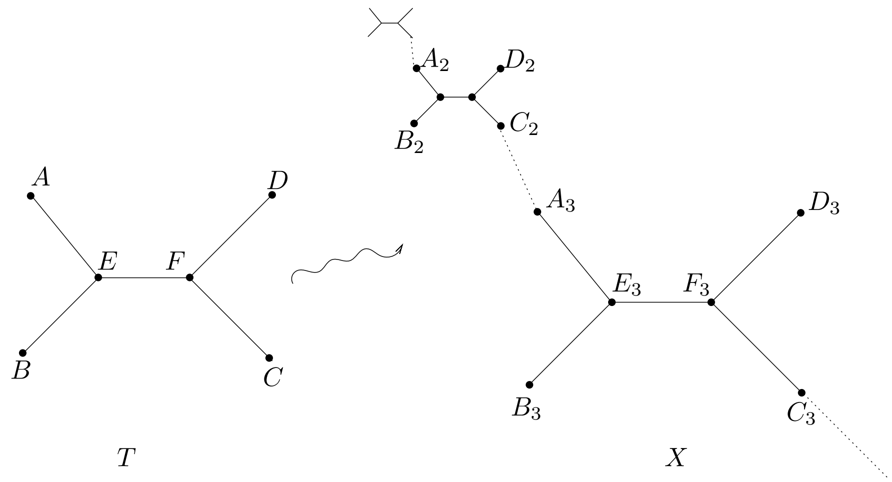


Figure 4.1: Building the space in Ex. 31 3.

Remark 32. *Example 31 3. is actually an example of a map which is power quasi-isometric as defined in [BS1] Def. 4.3.1. Our P-QI maps (Def. 30) are called* strongly power quasi-isometric *in [BS1] Def. 4.1.1. The difference is that P-QI maps as defined by Buyalo and Schroeder are maps which do preserve any* single *Gromov product in the usual "quasi-way", but they do not necessarily control the* difference of two Gromov products, *while our P-QI maps (or* strongly P-QI *in [BS1]) also preserve differences of Gromov products, which follows from the fact that $\frac{1}{2}(|xz|+|yw|-|xy|-|zw|) = (x|y)_z - (w|y)_z$. See also [BS1] Lemma 4.2.3 and Prop. 4.3.2.*

Proposition 33. *If $F : X \to Y$ is P-QI then X is Gromov hyperbolic if and only if $F(X) \subset Y$ is Gromov hyperbolic.*

Proof. $\Rightarrow$: Suppose the cross-difference triple $\{r, s, t\}$ of a quadruple of points in X is a δ-triple and F is (c, d)-P-QI. Assume w.l.o.g. $r \leq s \leq t$ and denote by r', s', t' the appropriate quantities w.r.t. the images under F. If $r' \leq s' \leq t'$ or $r' \leq t' \leq s'$ it is immediate that $\{r', s', t'\}$ is a $(c\delta + d)$-triple. It only remains to check the case $t' \leq r' < s'$. But since $0 \leq r' - t' \leq c(r - t) + d \leq d$, $\{r', s', t'\}$ is a d-triple in this case.

$\Leftarrow$: Follows from the fact that a rough inverse of a P-QI map is also P-QI. $\square$

4.2 Quasimöbius and Quasisymmetric Maps

In Section 4.1 we described roughly isometric and (power)-quasi-isometric maps between Gromov hyperbolic spaces. In a rather explicit sense, the properties of maps between hyperbolic spaces are "exponentiated" to yield the appropriate properties for boundary maps. Roughly isometric maps will be related to *bilipschitz* and *bilipschitz-quasimoebius* maps and quasi-isometric maps will be related to so-called *power quasisymmetric* and *power quasimb̈ius* maps. These quasimb̈ius maps are characterized by how they control the *cross-ratio* of a quadruple of points.

Definition 34. *Let (Z, ρ) be a quasi-metric space. For any quadruple $Q = (x, y, z, w) \in Z^4 \setminus D$, where $D \subset Z^4$ is the subset where the same point appears three or four times, define the* cross-ratio *of Q, $cr(Q)$, by*

$$cr(Q) = \frac{\rho(x,z)\rho(y,w)}{\rho(x,y)\rho(z,w)}.$$

Remark 35. *If a point appears more than once in a quadruple Q, we define $cr(Q)$ via the following conventions (where distinct letters denote distinct points and ω denotes the infinitely remote point of Z, if it exists)*

$$\begin{aligned} cr(x,x,y,z) &:= \infty \qquad cr(x,y,x,z) := 0 \\ cr(x,x,x,y) &:= 1 \\ cr(x,y,z,\omega) &:= \frac{\rho(x,z)}{\rho(x,y)} \\ cr(x,y,\omega,\omega) = cr(x,x,\omega,\omega) &:= \infty \qquad cr(x,\omega,y,\omega) = cr(x,\omega,x,\omega) := 0 \end{aligned}$$

Definition 36. *If $\theta : [0, \infty) \to [0, \infty)$ is a homeomorphism, an injective map $f : Z \to Z'$ between quasi-metric spaces is called θ-quasimb̈ius (θ-QM) if*

$$\frac{1}{\theta(\frac{1}{cr(Q)})} \leq cr(f(Q)) \leq \theta(cr(Q)).$$

f is called power quasimöbius *(P-QM) if it is θ-QM for a θ of the form $\theta(t) = q \max\{t^{1/p}, t^p\}$. It is called* bilipschitz quasimöbius *(BL-QM) if θ can be taken linear, $\theta(t) = \lambda t$.*

Closely related to QM maps are *quasisymmetric* (QS) maps, which are the injective maps that preserve the ordinary ratio sr of a triple (x, y, z), $sr(x, y, z) := \rho(x, z)/\rho(x, y)$, in an analogous way.

Note that a non-constant quasisymmetric map is automatically injective. The same is true for quasimöbius maps.

We refer to [V] and [BS1] Ch. 5, for more information on quasimöbius and quasisymmetric maps. We just note that every quasisymmetric map is quasimöbius and that quasimöbius maps are homeomorphisms onto their images. In particular, they map complete spaces to complete spaces.

Remark 37. *In fact, the bilipschitz class of $\partial_\infty^{a,o} X$ does not depend on $o \in X$ and the quasimöbius class depends on neither of the parameters. Thus we may suppress one or both of them and just write $\partial_\infty^a X$, or $\partial_\infty X$. Whenever we do this it is to be understood that the statement holds for any admissible choice of the omitted parameter(s).*

To finish the chapter, we cite the mentioned theorem of Buyalo-Schroeder which says that it is not necessary to distinguish between quasi-isometric and P-QI maps in the setting of geodesic spaces. The proof is based on the stability of geodesics, Thm. 22.

Theorem 38 ([BS1] Thm. 4.4.1)**.** *If X, Y are geodesic Gromov hyperbolic metric spaces with hyperbolicity constants δ and δ' respectively and if $F : X \to Y$ is a (c, b)-quasi-isometric map. Then F is (c, d)-P-QI, where $d = d(c, b, \delta, \delta')$.* $\square$

Example 39 (Gromov hyperbolic groups)**.** *Consider a finitely generated group with a symmetric generating set $G =< S|R >$. Associated to this presentation is the Cayley graph Γ of G, whose vertices are the group elements and two vertices v, v' are joined by an edge exactly when $v = v's_i$ for some generator s_i. Define the length of every edge to be 1 and consider the induced length metric on Γ. Then Γ is clearly a geodesic metric space.*

If S' is a different (symmetric) generating set for G, then the word-lenghts l w.r.t S and l' w.r.t. S' are quasi-isometric to each other, which is easily seen by considering the longest and shortest elements of one generating set in the word-length of the other generating set. By Thm. 38, the Cayley graph Γ associated to S is thus a geodesic Gromov hyperbolic metric space if and only if the Cayley graph associated to Γ' is.

A group is called a Gromov hyperbolic group *if the Cayley graph associated to one (and hence any) generating set is a Gromov hyperbolic space.*

Gromov hyperbolic groups have played an important role in geometric group theory, geometric topology and algorithmics in recent years, see the references we listed in the Introduction.

Chapter 5

The Boundary at Infinity

To every Gromov hyperbolic metric space X one can associate a so-called *boundary at infinity*, $\partial_\infty X$, of X. This is a (quasi)metric space which encodes to a certain degree what the space X looks like on large scales. The idea and the construction of the space is in some sense analogous to the Tits boundary of Hadamard manifolds or CAT(0) spaces.

5.1 The Boundary as a Set

5.1.1 The Geodesic Boundary $\partial_g X$

To underscore the similarities to the theory of Tits boundaries in CAT(0)-spaces we first introduce the *geodesic boundary at infinity.* Even though this set will in general only be a subset of the "real" boundary at infinity defined in the next section, it turns out that for *proper* geodesic Gromov hyperbolic spaces the two definitions are equivalent.

Definition 40 (Asymptotic rays)**.** *Let X be a Gromov hyperbolic metric space and $\gamma, \gamma' : [0, \infty) \to X$ two geodesic rays. We say γ is* asymptotic to *γ', $\gamma \sim \gamma'$, if $\sup_t |\gamma(t)\gamma'(t)| < \infty$, or what is the same (triangle inequality!), $d_H(\gamma, \gamma') < \infty$.*

The relation to be asymptotic is obviously an equivalence relation.

Definition 41. *The* geodesic boundary at infinity *of a Gromov hyperbolic space X, $\partial_g X$, is the set of equivalence classes of asymptotic rays in X.*

Remark 42. *Clearly if a Gromov hyperbolic space X does not have a lot of geodesics, the geodesic boundary will typically be very small. For example, the Gromov hyperbolic space $\mathbb{Z}$ has no geodesic rays, thus $\partial_g X = \emptyset$. But even for geodesic spaces the geodesic rays in general do not capture the whole asymptotic geometry of X. We give an example in the next section.*

5.1.2 The Gromov Boundary $\partial_\infty X$

In the general construction of $\partial_\infty X$ the rays used for $\partial_g X$ are replaced with *sequences converging to infinity.*

Definition 43. *A sequence* $\{x_i\} \subset X$ *in a Gromov hyperbolic metric space* X *is said to* converge to infinity *if* $(x_i|x_j)_o \to \infty$ *for some* $o \in X$.

Note that $|(x|y)_o - (x|y)_{o'}| \leq |oo'|$, so the definition does not depend on the base point o.

Definition 44 (Asymptotic sequences)**.** *Two sequences* $\{x_i\}, \{y_i\}$ *in* X *are called* asymptotic*, or* equivalent*, if* $(x_i|y_i)_o \to \infty$ *for some, and hence any,* $o \in X$.

Because $\{(x_i|y_i)_o, (x_i|y_i)_o, (y_i|z_i)_o\}$ is a δ-triple for each i, the relation to be asymptotic is an equivalence relation among sequences converging to infinity.

Definition 45. *The* Gromov boundary at infinity*, or just* boundary at infinity*,* $\partial_\infty X$*, of a Gromov hyperbolic metric space* X *is the set of equivalence classes of sequences converging to infinity.*

Elements of $\partial_\infty X$ are usually denoted by lower case greek letters $\xi, \eta, \zeta, \ldots$.
We obviously have $\partial_g X \subset \partial_\infty X$. By an Arzelà-Ascoli argument, one can show that if X is proper and geodesic, then to every $\xi \in \partial_\infty X$ and every $o \in X$ there exists a ray γ from o to ξ. For non-proper spaces this argument does not work and in general $\partial_g X \neq \partial_\infty X$.

Example 46. *Consider the following metric graph* X*, cf. Fig. 5.1. Take the non-negative real half-line* $\mathbb{R}_{\geq 0}$*. Add for each* $k \geq 1$ *an edge of length* 1 *from* 0 *to* $1 + 1/2^k$*. From each of the endpoints* $1 + 1/2^k$ *of these edges, draw one edge of length* 1 *to* $2 + 1/2^{k-1}$ *for* $k \geq 2$*. Continue like this by drawing an edge of length* 1 *from* $n + 1/2^k$ *to* $(n+1) + 1/2^{k-1}$ *for each* $n \geq 2$ *and* $k \geq 2$.

X is a geodesic Gromov hyperbolic space, but it is impossible to define an infinite geodesic. In particular, $\partial_g X = \emptyset$*. But clearly,* $\partial_\infty X = \{\omega\}$ *is a one-point set.*

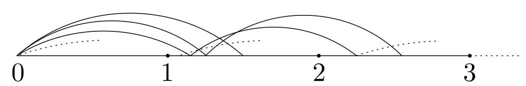


Figure 5.1: Geodesic hyperbolic space with no infinite geodesics.

5.2 The Boundary as a Quasimetric Space

We now introduce quasi-metrics on the set $\partial_\infty X$.

5.2.1 Gromov Product on the Boundary

We have seen that if X is a δ-hyperbolic space, then

$$\{(x|y)_o, (x|z)_o, (y|z)_o\}$$

is a δ-triple for all $x, y, z, o \in X$. We now extend the Gromov product to $\partial_\infty X$ as follows. Let $\xi, \xi' \in \partial_\infty X$ and $o \in X$. Set

$$(\xi|\xi')_o := \inf \liminf_{i\to\infty} (x_i|x_i')_o,$$

where the infimum is taken over all sequences $(x_i) \in \xi$, $(x_i') \in \xi'$. It is a fact ([BS1] Lemma 2.2.2(2)) that with this definition the δ-inequality extends to the boundary at infinity. That is,

$$\{(\xi|\xi')_o, (\xi|\xi'')_o, (\xi'|\xi'')_o\}$$

is a δ-triple for all $\xi, \xi', \xi'' \in \partial_\infty X$. Now example 12, 3 becomes clear. If X is a δ-hyperbolic space, $a > 1$, $o \in X$ and $(\cdot|\cdot)_o$ denotes the Gromov product with respect to the base point o, then $a^{-(\cdot|\cdot)_o}$ is an a^δ-quasi-metric on the set $\partial_\infty X$. Note that this quasi-metric is always bounded by 1, because the Gromov product is greater or equal to 0, $(\cdot|\cdot)_o \geq 0$.
Furthermore, we obtain the following properties.

Lemma 47 (Cf. e.g. [BS1] Lemma. 2.2.2)**.** *Let X be a δ-hyperbolic space, let $o \in X$ and $\xi, \xi' \in \partial_\infty X$. Then for arbitrary sequences $\{y_i\} \in \xi, \{y_i'\} \in \xi'$, we have*

$$(\xi|\xi')_o \leq \liminf_{i\to\infty} (y_i|y_i')_o \leq \limsup_{i\to\infty} (y_i|y_i')_o \leq (\xi|\xi')_o + 2\delta.$$

5.2.2 Boundary continuous Gromov hyperbolic spaces

A Gromov hyperbolic space is called *boundary continuous*, if the Gromov product extends continously to the boundary in the following way: if (x_i), (y_i) are sequences in X which converge to points $\overline{x}$, $\overline{y}$ in X or $\partial_\infty X$, then $(x_i|y_i)_o \to (\overline{x}|\overline{y})_o$ for all basepoints $o \in X$. For boundary continuous spaces one can define nicely Busemann functions. If $\omega \in \partial_\infty X$ and $o \in X$ a basepoint, then

$$b_{\omega,o}(x) = \lim_{i\to\infty} (|xw_i| - |w_i o|) \tag{5.1}$$

where $w_i \to \omega$ is the Busemannfunction of ω normalized to have the value 0 at the point $o \in X$. We have the formula:

$$b_{\omega,o}(x) = (\omega|o)_x - (\omega|x)_o \tag{5.2}$$

We also define form $\omega \in \partial_\infty X$ a base point $o \in X$ and x, y, z from X or $\partial_\infty X \setminus \{\omega\}$

$$(x|y)_{\omega,o} = (x|y)_o - (\omega|x)_o - (\omega|y)_o.$$

5.2.3 Busemann Functions and Inversions

In the previous section we have seen how to put a *bounded* quasi-metric on $\partial_\infty X$. Now in the classical setting for the hyperbolic plane $\mathbb{H}^2$, the boundary comes in two different shapes, once as S^1, the boundary of the unit disk model, and once as $\mathbb{R} \cup \{\infty\}$, the boundary of the upper half plane model. The two spaces, S^1 and $\mathbb{R} \cup \{\infty\}$ are related via the stereographic projection, a Moebius map. The quasi-metrics $a^{-(\cdot|\cdot)_o}$ we introduced in §5.2.1 are the analogs of S^1. Our goal in this section is to introduce a second type of quasi-metrics on the boundary which should play the role that $\mathbb{R} \cup \{\infty\}$ does in the classical case.

The crucial point is to realize that stereographic projection $S^1 \to \mathbb{R} \cup \{\infty\}$ is in fact an example of an inversion in a circle, namely the circle that is centered at the north pole $(0, 1)$ and has radius $\sqrt{2}$, cf. [BS1] §5.3.2 for details. And there is a formula for how the distance changes when one applies such an inversion. For example, invert $\mathbb{R}^2 \cup \{\infty\}$ in the unit circle. Denote the inversion by ι. Then if $(r_1, \varphi_1), (r_2, \varphi_2)$ are polar coordinates of two points we get

$$\begin{aligned} ||\iota(r_1, \varphi_1) - \iota(r_2, \varphi_2)|| &= ||(1/r_1, \varphi_1) - (1/r_2, \varphi_2)|| \\ &= \sqrt{1/r_1^2 + 1/r_2^2 - 2/(r_1 r_2)\cos(\varphi_1 - \varphi_2)}, \end{aligned}$$

while

$$||(r_1, \varphi_1) - (r_2, \varphi_2)|| = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\varphi_1 - \varphi_2)}.$$

In other words,

$$||\iota(r_1, \varphi_1) - \iota(r_2, \varphi_2)|| = \frac{||(r_1, \varphi_1) - (r_2, \varphi_2)||}{||(r_1, \varphi_1)|| \cdot ||(r_2, \varphi_2)||}.$$

In general, one can show (cf. [BS1] §5.3.2) that if the inversion circle is centered at the point (r_o, φ_o) and has radius r, then

$$||\iota(r_1, \varphi_1) - \iota(r_2, \varphi_2)|| = \frac{r^2 ||(r_1, \varphi_1) - (r_2, \varphi_2)||}{||(r_1, \varphi_1) - (r_o, \varphi_o)|| \cdot ||(r_2, \varphi_2) - (r_o, \varphi_o)||}.$$

This leads us to define the following.

Definition 48 (Inverted quasi-metric). *If (Z, ρ) is a quasi-metric space and $o \in Z \setminus \{\infty\}$, $r > 0$, the* inversion ρ' of ρ at center o with radius r *is defined by*

$$\rho'(u, v) := \frac{r^2 \rho(u, v)}{\rho(u, o)\rho(v, o)}.$$

It is not difficult to show (see [BS1], proof of Prop 5.3.6) that ρ' is a K^2-quasi-metric whenever ρ is a K-quasi-metric.

If now X is some Gromov hyperbolic space X, $o \in X$ and $\omega \in \partial_\infty X$, define the function

$$\begin{aligned} b_{o,\omega} : X &\to \mathbb{R} \\ x &\mapsto |ox| - 2(\omega|x)_o, \end{aligned}$$

and consider the following *Gromov product based at ω*:

$$(x|y)_{b_{o,\omega}} := \frac{1}{2}(b_{o,\omega}(x) + b_{o,\omega}(y) - |xy|).$$

A trivial computation shows $(x|y)_{b_{o,\omega}} = (x|y)_o - (\omega|x)_o - (\omega|y)_o$. This suggests that $a^{-(\cdot|\cdot)_{b_{o,\omega}}}$ is the inversion of $a^{-(\cdot|\cdot)_o}$ with inversion center ω and radius 1. Of course, for this statement to make sense we first have to extend $(\cdot|\cdot)_{b_{o,\omega}}$ to $\partial_\infty X$. This is done just as in the case of $(\cdot|\cdot)_o$, namely

$$(\xi|\xi')_{b_{o,\omega}} := \inf \liminf_{i\to\infty} (x_i|x_i')_{b_{o,\omega}},$$

where the infimum is taken over all sequences $(x_i) \in \xi, (x_i') \in \xi'$.

The function $b_{o,\omega}$ is an example of a *Busemann function*, which are well-known in the theory of non-positively curved manifolds. The complete set of Busemann functions for a Gromov hyperbolic space is defined as follows.

Definition 49 (Cf. e.g. [BS1] Def. 3.1.3)**.** *Let X be a δ-hyperbolic space and $\omega \in \partial_\infty X$ fixed. The set $\mathcal{B}(\omega)$ of all* Busemann functions based at ω *consists of all those functions $b : X \to \mathbb{R}$ for which there exists $o \in X$ and a constant $c \in \mathbb{R}$ such that $b \doteq_{2\delta} b_{\omega,o} + c$.*

The δ-inequality carries over to these Gromov products as well.

Proposition 50 ([BS1] Lemma 3.2.4(2))**.** *For X a δ-hyperbolic space and $\xi, \eta, \zeta, \omega \in \partial_\infty X$ arbitrary, the numbers $(\xi|\eta)_b$, $(\xi|\zeta)_b$, $(\eta|\zeta)_b$ form a 22δ-triple for any $b \in \mathcal{B}(\omega)$.* □

This shows that the quasi-metric $a^{-(\cdot|\cdot)_{b_{o,\omega}}}$ on $\partial_\infty X$ is bilipschitz equivalent to the quasi-metric obtained by inverting $a^{-(\cdot|\cdot)_o}$ in ω and with radius 1.

We are now in a position to define the most general form of the boundary at infinity.

Definition 51. *Let X be a Gromov hyperbolic space, $a > 1$, $o \in X$, $b \in \mathcal{B}(\omega)$ for some $\omega \in \partial_\infty X$.*

The symbol $\partial_\infty^{a,o} X$ denotes the quasi-metric space $(\partial_\infty X, a^{-(\cdot|\cdot)_o})$.

The symbol $\partial_\infty^{a,b}$ denotes the quasi-metric space $(\partial_\infty X, a^{-(\cdot|\cdot)_b})$.

It is well-known that the boundary at infinity of a Gromov hyperbolic space is a complete quasi-metric space. For a proof, see e.g. [BS1] Prop. 6.2 (the proof carries over verbatim to the quasi-metric setting).

5.3 Induced Maps Between Boundaries

If a map $F : X \to X'$ between Gromov hyperbolic spaces maps sequences going to infinity in X to sequences going to infinity in X' and equivalent sequences to equivalent sequences, then F induces a map between boundaries, which we denote $\partial_\infty F : \partial_\infty X \to \partial_\infty X'$.

For example, every roughly isometric map $F : X \to X'$ induces an injection $\partial_\infty F : \partial_\infty X \to \partial_\infty X'$. A quasi-isometric map $F : X \to X'$ between *geodesic* hyperbolic spaces induces a boundary map by the stability of geodesics (cf. [BH] Thm. III.H.1.7). However, the map $F : \{10^i | i \in \mathbb{N}\} \to \mathbb{R}$, $F(10^i) := (-1)^i 10^i$ is quasi-isometric, but does not induce a boundary map in any reasonable sense. This is another reason why quasi-isometric maps are in general not the right maps to look at in the setting of hyperbolic metric spaces (we have already seen that quasi-isometric maps need not preserve hyperbolicity). In contrast, any P-QI map *does* induce a map between associated boundaries. This follows from $cd(x, y, o, o) = (x|y)_o$.

Remark 52. *In fact, also the map of Ex. 31 3 induces a canonical boundary map, because it preserves Gromov products in a quasi-way. It is not P-QI according to our definition, however. The reason we avoid maps of this type and consider only P-QI maps is that it seems to be impossible to recover non-P-QI maps from the boundary maps they induce, much in contrast to P-QI maps.*

We quote now well-known results about maps between Gromov hyperbolic spaces that induce maps between the boundaries associated to the spaces. Recall that omitting values for base points or Busemann functions in $\partial_\infty^{a,o}, \partial_\infty^{a,b}$ means that the statement is valid for any choice of them.

Theorem 53 (Cf. e.g. [BS1] Thm. 5.2.10)**.** *If $F : X \to X'$ is a roughly isometric map between Gromov hyperbolic spaces, then for each $a > 1$ F induces a bilipschitz-quasimoebius map $\partial_\infty F : \partial_\infty^a X \to \partial_\infty^a X'$.*

If $F : X \to X'$ is a power quasi-isometric map between Gromov hyperbolic spaces, then F induces a power quasimoebius map between associated boundaries, $\partial_\infty F : \partial_\infty^a X \to \partial_\infty^{a'} X'$, where $a, a' > 1$. □

5.4 Asymptotic Curvature and Visual Metrics

This section is independent of the major results of this thesis and skipping it will have no ill effects on the understanding of the rest of this work. Our purpose in this section is to give an example of a visual geodesic Gromov hyperbolic space with asymptotic curvature -1 such that $e^{-(\cdot|\cdot)_o}$ is *not* bilipschitz equivalent to any metric on $\partial_\infty X$.

As mentioned in the previous section, for every Gromov hyperbolic space X and $o \in X$, $a^{-(\cdot|\cdot)_o}$ becomes bilipschitz equivalent to an honest metric on $\partial_\infty X$ when a is close enough to 1. The notion of *asymptotic curvature* of a Gromov hyperbolic space was introduced by Bonk and Foertsch in [BF] and, at least for visual spaces, it quantifies just how close to 1 a has to be taken. It is defined as follows.

Definition 54. *Let X be a metric space and $\kappa \in [-\infty, 0)$. We say that X has an* asymptotic curvature bound *κ, or X* is *$AC_u(\kappa)$, if there exists $p \in X$ and a constant $c \geq 0$ such that for all $z, z' \in X$ and all chains $z = x_0, x_1, \dots, x_n = z'$ in X we have*

$$(z|z')_p \geq \min_i (x_{i-1}|x_i)_p - \frac{1}{\sqrt{-\kappa}} \log n - c,$$

where $\frac{1}{\sqrt{\infty}} := 0$.

The asymptotic curvature *of X, $K_u(X)$, is then given by*

$$K_u(X) := \inf\{\kappa \mid X \textit{ is an } AC_u(\kappa)\textit{-space}\}.$$

If we parameterize a by e^ϵ with $\epsilon > 0$, we have the following connection between the asymptotic curvature of X and visual metrics on $\partial_\infty X$.

Theorem 55 ([BF] Thm. 1.5)**.** *Let X be a visual Gromov hyperbolic metric space. Then*

$$K_u(X) = -b^2,$$

where

$$b := \sup\{\epsilon > 0 \mid \textit{there exists a visual metric on } \partial_\infty X \textit{ with parameter } \epsilon\}. \square$$

Remark 56. *Julian Jordi in his Ph.D thesis gave a construction of a visual Gromov hyperbolic spaces where the supremum b is not attained, namely a visual Gromov hyperbolic space* X *with* $K_u(X) = -1$ *but such that* $e^{-(\cdot|\cdot)_o}$ *is* not *bilipschitz to any metric on* $\partial_\infty X$.

Chapter 6

Möbius Structures and Asymptotically PT_κ space

6.1 Möbius Structures

Let Z be a set which contains at least two points. An *extended metric* on Z is a map $d : Z \times Z \to [0,\infty]$, such that there exists a set $\Omega(d) \subset Z$ with cardinality $\#\Omega(d) \in \{0,1\}$, such that d restricted to the set $Z \setminus \Omega(d)$ is a metric (taking only values in $[0,\infty)$) and such that $d(z,\omega) = \infty$ for all $z \in Z \setminus \Omega(d)$, $\omega \in \Omega(d)$. Furthermore $d(\omega,\omega) = 0$.
If $\Omega(d)$ is not empty, we call the unique $\omega \in \Omega(d)$ simply *the point at infinity* of (Z,d). We write Z_ω for the set $Z \setminus \{\omega\}$.

The topology considered on (Z,d) is the topology with the basis consisting of all open distance balls $B_r(z)$ around points in $z \in Z_\omega$ and the complements D^C of all closed distance balls $D = \overline{B}_r(z)$.

We call an extended metric space (Z,d) *complete*, if first every Cauchy sequence in Z_ω converges and secondly if the infinitely remote point ω exists in case that Z_ω is unbounded. For example the real line $(\mathbb{R},d)$, with its standard metric is *not* complete (as extended metric space), while $(\mathbb{R}\cup\{\infty\},d)$ is complete.

We say that a quadruple $(x,y,z,w) \in Z^4$ is *admissible*, if no entry occurs three or four times in the quadruple. We denote with $Q \subset Z^4$ the set of admissible quadruples. We define the *cross ratio triple* as the map $\mathrm{crt} : Q \to \Sigma \subset \mathbb{R}P^2$ which maps admissible quadruples to points in the real projective plane defined by

$$\mathrm{crt}(x,y,z,w) = (d(x,y)d(z,w) : d(x,z)d(y,w) : d(x,w)d(y,z)),$$

here Σ is the subset of points $(a : b : c) \in \mathbb{R}P^2$, where all entries a,b,c are nonnegative or all entries are nonpositive.

We use the standard conventions for the calculation with ∞. If ∞ occurs once in Q, say $w = \infty$, then $\mathrm{crt}(x,y,z,\infty) = (d(x,y) : d(x,z) : d(y,z))$. If ∞ occurs twice , say $z = w = \infty$ then $\mathrm{crt}(x,y,\infty,\infty) = (0 : 1 : 1)$.

Similar as for the classical cross ratio there are six possible definitions by permuting the entries and we choose the above one.

A map $f : Z \to Z'$ between two extended metric spaces is called *Möbius*, if f is injective and for all admissible quadruples (x, y, z, w) of X,

$$\operatorname{crt}(f(x), f(y), f(z), f(w)) = \operatorname{crt}(x, y, z, w).$$

Möbius maps are continuous.

Two extended metric spaces (Z, d) and (Z, d') are *Möbius equivalent*, if there exists a bijective Möbius map $f : Z \to Z$. In this case also f^{-1} is a Möbius map and f is in particular a homeomorphism.
We say that two extended metrics d and d' on the same set Z are *Möbius equivalent*, if the identity map $\mathrm{id} : (Z, d) \to (Z, d')$ is a Möbius map. Möbius equivalent metrics define the same topology on Z. It is also not difficult to check that for Möbius equivalent metrics d and d' the space (Z, d) is complete if and only if (Z, d') is complete.

The Möbius equivalence of metrics of metrics on a given set Z is clearly an equivalence relation. A *Möbius structure* $\mathcal{M}$ on Z is an equivalence class of extended metrics on Z.

A pair $(Z, \mathcal{M})$ of a set Z together with a Möbius structure $\mathcal{M}$ on Z is called a *Möbius space*. A Möbius structure well defines a topology on Z, thus a Möbius space is in particular a topological space. Since completeness is also a Möbius invariant we can speak about *complete* Möbius structures.

In general two metrics in $\mathcal{M}$ can look very different. However if two metrics have the same remote point at infinity, then they are homothetic (see [FS3]):

Lemma 57. *Let $\mathcal{M}$ be a Möbius structure on a set X, and let $d, d' \in \mathcal{M}$, such that $\omega \in X$ is the remote point of d and of d'. Then there exists $\lambda > 0$, such that $d'(x, y) = \lambda d(x, y)$ for all $x, y \in X$.*

An extended metric space (Z, d) is called a *Ptolemy space*, if for all quadruples of points $\{x, y, z, w\} \in Z^4$ the *Ptolemy inequality* holds

$$d(x, y)\, d(z, w) \ \le\ d(x, z)\, d(y, w) \ +\ d(x, w)\, d(y, z)$$

We can reformulate this condition in terms of the cross ratio triple. Let $\Delta \subset \Sigma$ be the set of points $(a : b : c) \in \Sigma$, such that the entries a, b, c satisfy the triangle inequality. This is obviously well defined.

Then an extended space is Ptolemy, if $\operatorname{crt}(x, y, z, w) \in \Delta$ for all allowed quadruples Q.

This description shows that the Ptolemy property is Möbius invariant and thus a property of the Möbius structure $\mathcal{M}$.

The importance of the Ptolemy property comes from the following fact (sse e.g. [FS3]):

Theorem 58. *A Möbius structure $\mathcal{M}$ on a set Z is Ptolemy, if and only if for all $\omega \in Z$ there exists $d_\omega \in \mathcal{M}$ with $\Omega(d_\omega) = \{\omega\}$.*

The metric d_ω can be obtained by metric involution. If d is a metric on Z then

$$d_\omega(z, z') = \frac{d(z, z')}{d(\omega, z) d(\omega, z')}$$

gives the required metric.

6.2 PT_κ spaces

6.2.1 $\mathrm{CAT}(\kappa)$ spaces

One way of characterizing geodesic $\mathrm{CAT}(\kappa)$ spaces, $\kappa \in \mathbb{R}$, is by using the so-called 4-point condition. Suppose $x_i \in X$ and $\bar{x}_i \in M^2$ for $0 \le i \le 4$, with $x_0 = x_4$ and $\bar{x}_0 = \bar{x}_4$. We say that $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ is a subembedding of (x_1, x_2, x_3, x_4) in M^2 if

$$d(x_i, x_{i-1}) = |\bar{x}_i - \bar{x}_{i-1}|, \quad 1 \le i \le 4,$$

$$d(x_1, x_3) \le |\bar{x}_1 - \bar{x}_3| \quad \text{and} \quad d(x_2, x_4) \le |\bar{x}_2 - \bar{x}_4|.$$

The metric space (X, d) satisfies the 4-point condition, if every 4-tuple in X has a subembedding in M^2_κ. When X is geodesic, this turns out to be equivalent to X being $\mathrm{CAT}(\kappa)$.

6.2.2 PT_κ inequality

A metric space (X, d) is called a PT_κ-space, if for points $x_1, x_2, x_3, x_4 \in X$, we have

$$\mathrm{sn}_\kappa(\frac{\rho_{1,3}}{2})\,\mathrm{sn}_\kappa(\frac{\rho_{2,4}}{2}) \;\le\; \mathrm{sn}_\kappa(\frac{\rho_{1,2}}{2})\,\mathrm{sn}_\kappa(\frac{\rho_{3,4}}{2}) \;+\; \mathrm{sn}_\kappa(\frac{\rho_{1,4}}{2})\,\mathrm{sn}_\kappa(\frac{\rho_{2,3}}{2}) \tag{6.1}$$

where $\rho_{i,j} = d(x_i, x_j)$ and sn_κ is the function

$$\mathrm{sn}_\kappa(x) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} x) & \text{if } \kappa > 0, \\ x & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} x) & \text{if } \kappa < 0. \end{cases}$$

In the case that $\kappa > 0$ we assume in addition that the diameter is bounded by $\frac{\pi}{\sqrt{\kappa}}$.

It is well known that the standard space forms M^n_κ of constant curvature κ satisfy the PT_κ inequality. For the euclidean space this is the classical Ptolemy inequality and for the other spaces it is proved in [**?**]. By comparison we obtain the result also for $\mathrm{CAT}(\kappa)$-spaces.

A model representing the elliptic space can be obtained by means of stereographic projection. Let E^n represent $\mathbb{R}^n \cup \{\infty\}$, that is, n-dimensional real space extended by a single point at infinity. We may define a metric, the chordal metric, on E^n by

$$\delta(u, v) = \frac{2||u - v||}{\sqrt{(1 + ||u||^2)(1 + ||v||^2)}},$$

where u and v are any two vectors in $\mathbb{R}^n$ and $||\cdot||$ is the usual Euclidean norm. We also define

$$\delta(u, \infty) = \delta(\infty, u) = \frac{2}{\sqrt{1 + ||u||^2}}.$$

The result is a metric space on E^n, which represents the distance along a chord of the corresponding points on the hyperspherical model, to which it maps bijectively by stereographic projection. We obtain a model of spherical geometry if we use the metric

$$d(u,v) = 2\arcsin(\frac{\delta(u,v)}{2}).$$

We can easily verify that the $(\mathbb{S}^3, d)$ satisfies PT_1 inequality.

Proposition 59. *Every* $\mathrm{CAT}(\kappa)$ *space satisfies the* PT_κ *inequality.*

Proof. A $\mathrm{CAT}(\kappa)$ spaces, $\kappa \in \mathbb{R}$, can be characterized by a 4-point condition, [BH]. Suppose $x_i \in X$ for $0 \le i \le 4$, with $x_0 = x_4$, and $x_0 = x_4$, there exist four points $\bar{x}_i \in M_\kappa^2$ with $\bar{x}_0 = \bar{x}_4$ such that

$$d(x_i, x_{i-1}) = |\bar{x}_i - \bar{x}_{i-1}|, 1 \le i \le 4,$$

$$d(x_1, x_3) \le |\bar{x}_1 - \bar{x}_3| \quad \text{and} \quad d(x_2, x_4) \le |\bar{x}_2 - \bar{x}_4|.$$

Since M_κ^2 satisfy the PT_κ inequality the result follows. □

Remark 60. *The space X is called* distance convex *if for all $p \in X$ the distance function $d_p = |\cdot\ p|$ to the point p is convex. It is called* strictly distance convex, *if the functions $t \mapsto (d_p \circ c)(t)$ are strictly convex whenever $c : I \to X$ is a geodesic with $|c(t)\, c(s)| > ||p\, c(t)| - |p\, c(s)||$ for all $s, t \in I$, i.e., neither $c(t)$ and $c(s)$ being on a geodesic from p to the other.*

Here are some properties for PT_κ geodesic spaces

Proposition 61. *If X is a* PT_κ *geodesic metric space, then every local geodesic is globally minimizing. Moreover, if $\kappa < 0$, then X is also strictly convex and hence geodesics are unique.*

Proof. Without loss of generality, we only prove it for $\kappa = 1$. Assume the contrary, there is a local geodesic $c : [0, b] \to X$, such that there exists $a \in (0, b)$ such that for $p = c(0)$ we have $|pc(a-\epsilon)| = a-\epsilon$ for all $\epsilon \ge 0$ but $|pc(a+\epsilon)| < a+\epsilon$ for $\epsilon > 0$. The PT_κ inequality shows that

$$\sin\frac{(a-\epsilon)}{2}\sin\frac{\epsilon}{2} + \sin\frac{(|pc(a+\epsilon)|)}{2}\sin\frac{\epsilon}{2} \ge \sin\frac{a}{2}\sin\epsilon$$

Hence, $|pc(a+\epsilon)| \ge a + \epsilon$. Contradiction!

If $\kappa < 0$, and given x_0, x_1 there are two geodesics connecting them saying γ_0 and γ_1. Now we will show that $d(\gamma_0(t), \gamma_1(t)) = 0$ for all $t \in [0, 1]$. If not, assume there exsits a point $t_0 \in [0, 1]$ such that $d(\gamma_0(t_0), \gamma_1(t_0)) > 0$. Choose the midpoint z between $\gamma_0(t_0)$ and $\gamma_1(t_0)$, and apply the PT_κ inequality for $x_0, \gamma_0(t_0), \gamma_1(t_0), z$ and $x_1, \gamma_0(t_0), \gamma_1(t_0), z$, We obtain

$$\sinh\frac{\sqrt{-\kappa}}{2}t_0 > \sinh\frac{\sqrt{-\kappa}}{2}d(x_0, z),\ \sinh(\frac{\sqrt{-\kappa}}{2}(1-t_0)) > \sinh(\frac{\sqrt{-\kappa}}{2}d(x_1, z))$$

Contradiction! Similar to prove the strict convexity. □

Corollary 62. *Let X be a complete* PT_κ *geodesic space with $\kappa < 0$, and $A \subset X$ be a closed and convex subset. Then there exists a continuous projection $\pi : X \to A$.*

Proof. Uniqueness is directly from the strict convexity. Now we just have to show the existence. Suppose $x \in X$ and let z_n be a sequence of points in A with $\lim_n d(z_n, x) = \inf_{z \in A} d(x, A)$. Denote $z_{n,k}$ be the midpoint between z_n and z_k. Applying the PT_κ inequality for $x, z_n, z_k, z_{n,k}$, we obtain

$$\begin{aligned} \sinh(\frac{\sqrt{-\kappa}}{2} d(x, z_n)) + \sinh(\frac{\sqrt{-\kappa}}{2} d(x, z_k)) \\ \geq 2 \sinh(\frac{\sqrt{-\kappa}}{2} d(x, z_{n,k})) \cosh(\frac{\sqrt{-\kappa}}{4} d(z_n, z_k)) \end{aligned} \quad (6.2)$$

Then for $n, k \to \infty$, $d(z_n, z_k) \to 0$, i.e. (z_n) is a Cauchy sequence and therefore there exists $z^* = \lim_{n \to \infty} z_n \in A$ since A is complete. $\square$

Proposition 63. *Consider a complete* PT_κ *geodesic metric space if it is also a topological manifold, then it is geodesic extendable.*

Proposition 64. *Normed real vector spaces are not* PT_κ *for any* $\kappa < 0$.

Proof. Suppose V is a normed real vector space and also PT_κ for some $\kappa < 0$. Then we know that $T_x V = V$ for some $x \in V$. Hence V is a normed vector space and also PT_0. From [Sch], we obtain that V is a Hilbert space. Hence can not be Gromov hyperbolic. This is the contradiction! $\square$

6.2.3 Asymptotic PT_κ inequality for $\kappa < 0$

One obtaines the asymptotic PT_κ property (for $\kappa < 0$) be weakening equation the PT_κ inequality and allowing some error term. Instead of equation (6.1) we require that for some universal $\delta \geq 0$ we have

$$\begin{aligned} \mathrm{sn}_\kappa(\frac{\rho_{1,3}}{2})\, \mathrm{sn}_\kappa(\frac{\rho_{2,4}}{2}) \quad \leq \\ (\mathrm{sn}_\kappa(\frac{\rho_{1,2}}{2}) + \delta)(\mathrm{sn}_\kappa(\frac{\rho_{3,4}}{2}) + \delta) \; + \; (\mathrm{sn}_\kappa(\frac{\rho_{1,4}}{2}) + \delta)(\mathrm{sn}_\kappa(\frac{\rho_{2,3}}{2}) + \delta) \end{aligned}$$

It is more convenient to formulate this condition using the exponential function. It is easy to check that these conditions are equivalent.

Definition 65. *A metric space is called asymptotic* PT_κ *for some* $\kappa < 0$*, if there exists some* $\delta \geq 0$ *such that for all quadruples* $x_1, x_2, x_3, x_4 \in X$ *we have*

$$e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,3}+\rho_{2,4})} \leq e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,2}+\rho_{3,4})} + e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,4}+\rho_{2,3})} + \delta e^{\frac{\sqrt{-\kappa}}{2}\rho}$$

Here $\rho_{i,j} = d(x_i, x_j)$ *and* $\rho = \max_{i,j} \rho_{i,j}$.

Remark 66. *The asymptotic* PT_κ *condition is a strong curvature condition. It implies e.g. that* X *does not contain flat strips: if a space contains a flat strip of width* $a > 0$*, then it contains quadruples with* $\rho_{1,3} = \rho_{2,4} = \sqrt{t^2 + a^2}$*,* $\rho_{1,2} = \rho_{3,4} = t$ *and* $\rho_{2,3} = \rho_{1,4} = a$*. These quadruples do not satisfy the asymptotic* PT_κ *inequality for fixed* $\kappa < 0$*,* $\delta \geq 0$ *and* $t \to \infty$.

Proposition 67. *Let* $0 > \kappa' > \kappa$*. If* X *is asymptotic* PT_κ*, then* X *is asymptotic* $\mathrm{PT}_{\kappa'}$.

Proof. From the aymptotic PT_κ inequality, we obtain that

$$e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,3}+\rho_{2,4})} \leq e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,2}+\rho_{3,4})} + e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,4}+\rho_{2,3})} + \delta e^{\frac{\sqrt{-\kappa}}{2}\rho}$$

Here $\rho = \max_{i,j}\rho_{i,j}$ for $i,j = 1,2,3,4$.
Since we know that for $0 \leq x \leq 1$

$$(a+b)^x \leq a^x + b^x, a > 0, b > 0.$$

Hence

$$\begin{aligned} e^{\frac{\sqrt{-\kappa'}}{2}(\rho_{1,3}+\rho_{2,4}} &= (e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,3}+\rho_{2,4})})^{\sqrt{\frac{-\kappa'}{-\kappa}}} \\ &\leq (e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,2}+\rho_{3,4})} + e^{\frac{\sqrt{-\kappa}}{2}(\rho_{1,4}+\rho_{2,3})} + \delta e^{\frac{\sqrt{-\kappa}}{2}\rho})^{\sqrt{\frac{-\kappa'}{-\kappa}}} \\ &\leq e^{\frac{\sqrt{-\kappa'}}{2}(\rho_{1,2}+\rho_{3,4})} + e^{\frac{\sqrt{-\kappa'}}{2}(\rho_{1,4}+\rho_{2,3})} + \delta' e^{\frac{\sqrt{-\kappa'}}{2}\rho} \end{aligned}$$

It satisfies the asymptotic $\mathrm{PT}_{\kappa'}$ inequality. □

By scaling an asymptotic PT_κ space with the factor $\frac{1}{\sqrt{-\kappa}}$ we obtain a PT_{-1} space. Therefore we will discuss in the sequel only PT_{-1} spaces.

Proposition 68. *An asymptotic* PT_{-1} *metric space is a Gromov hyperbolic space.*

Proof. The asymptotic PT_{-1} inequality is

$$e^{\frac{1}{2}(\rho_{1,3}+\rho_{2,4})} \leq e^{\frac{1}{2}(\rho_{1,2}+\rho_{3,4})} + e^{\frac{1}{2}(\rho_{1,4}+\rho_{2,3})} + \delta e^{\frac{1}{2}\rho}$$

Using the triangle inequality, we see

$$\rho \leq \max\{\rho_{1,2}+\rho_{3,4}, \rho_{1,4}+\rho_{2,3}\}$$

which then implies

$$e^{\frac{1}{2}(\rho_{1,3}+\rho_{2,4})} \leq (\delta+1)(e^{\frac{1}{2}(\rho_{1,2}+\rho_{3,4})} + e^{\frac{1}{2}(\rho_{1,4}+\rho_{2,3})})$$

and hence

$$\rho_{1,3}+\rho_{2,4} \leq \max\{\rho_{1,2}+\rho_{3,4}, \rho_{1,4}+\rho_{2,3}\} + \delta'.$$

Thus X is a Gromov hyperbolic space.

□

Lemma 69. *Let X be an asymptotic* PT_{-1} *space. Let (x_i) , (x'_i) and (y_i) be sequences in X satisfing*

$$\lim_{i\to\infty}(x_i|x'_i)_o = \infty, \quad \lim_{i\to\infty}(x_i|y_i)_o = a, \quad o \in X.$$

Then

$$\lim_{i\to\infty}(x'_i|y_i)_o = a$$

Proof. From the asymptotic PT_{-1} inequality, we obtain

$$\begin{aligned} e^{\frac{1}{2}(|x_i'y_i|+|ox_i|)} - e^{\frac{1}{2}(|oy_i|+|x_ix_i'|)} - \delta e^{\frac{1}{2}\rho_i} &\leq e^{\frac{1}{2}(|ox_i'|+|x_iy_i|)} \\ &\leq e^{\frac{1}{2}(|oy_i|+|x_ix_i'|)} + e^{\frac{1}{2}(|x_i'y_i|+|ox_i|)} + \delta e^{\frac{1}{2}\rho_i}, \end{aligned}$$

where $\rho_i = \max\{|ox_i|, |ox_i'|, |oy|, |x_ix_i'|, |x_iy_i|, |x_i'y_i|\}$.
Dividing both sides by $e^{\frac{1}{2}(|ox_i|+|ox_i'|+|oy_i|)}$, we obtain

$$e^{-(x_i'|y_i)_o} - e^{-(x_i|x_i')_o} - E_i \leq e^{-(x_i|y_i)_o} \leq e^{-(x_i'|y_i)_o} + e^{-(x_i|x_i')_o} + E_i,$$

where $E_i = \delta e^{\frac{1}{2}}(\rho_i - |ox_i| - |ox_i'| - |oy_i|)$. Note that by triangle inequalities

$$|ox_i| + |ox_i'| + |oy_i| - \rho_i \ \geq \min\{|ox_i|, |ox_i'|, 2(x_i|x_i')_o\},$$

and hence $E_i \to 0$ by our assumptions. Taking the limit, we obtain

$$\lim_{i\to\infty}(x_i'|y_i)_o = \lim_{i\to\infty}(x_i|y_i)_o = a.$$

□

As an immediate consequence we get

Corollary 70. *An asymptotic* PT_{-1} *space is boundary continuous.*

Theorem 71. *Let X be an asymptotic* PT_{-1} *metric space and $o \in X$, then*

$$\rho_o(x,y) = e^{-(x|y)_o}, \ \ x, y \in \partial_\infty X$$

is a metric on $\partial_\infty X$ which is PT_0.

Proof. First, we show that ρ_o is a metric on $\partial_\infty X$. For given three points $x, y, z \in \partial_\infty X$, choose sequences $(x_i) \in x, (y_i) \in y, (z_i) \in z$. By boundary continuity we have $(x|z)_o = \lim_{i\to\infty}(x_i|z_i)_o$. Then

$$e^{-(x|z)_o} = \lim_{i\to\infty} e^{\frac{1}{2}(|x_iz_i|-|x_io|-|z_io|)} = \lim_{i\to\infty} e^{-\frac{1}{2}(|x_io|+|y_io|+|z_io|)} e^{\frac{1}{2}(|x_iz_i|+|oy_i|)}$$

From the asymptotic PT_{-1} inequality, we have

$$e^{\frac{1}{2}(|x_iz_i|+|oy_i|)} \quad \leq \quad e^{\frac{1}{2}(|y_iz_i|+|ox_i|)} + e^{\frac{1}{2}(|x_iy_i|+|oz_i|)} + \delta e^{\frac{1}{2}\rho_i}$$

where $\rho_i = \max\{|ox_i|, |oy_i|, |oz_i|, |x_iy_i|, |x_iz_i|, |y_iz_i|\}$. Thus

$$e^{-(x|z)_o} \leq \lim_{i\to\infty} e^{\frac{1}{2}(|x_iy_i|-|ox_i|-|oy_i|)} + \lim_{i\to\infty} e^{\frac{1}{2}(|y_iz_i|-|oy_i|-|oz_i|)} + E_i,$$

where $E_i = \delta e^{\frac{1}{2}(\rho_i - |ox_i| - |oy_i| - |oz_i|}$. Again we easily check that $E_i \to 0$ and we obtain in the limit the triangle inequality for ρ_o.

We use the similar argument to show that ρ_o satisfies the Ptolemy inequality i.e.

$$e^{-(x|z)_o-(y|w)_o} \leq e^{-(x|y)_o-(z|w)_o} + e^{-(y|z)_o-(x|w)_o}.$$

Choose sequences $(x_i) \in x, (y_i) \in y, (z_i) \in z, (w_i) \in w$. Since we have

$$\begin{aligned} e^{-(x_i|z_i)_o-(y_i|w_i)_o} &= e^{-\frac{1}{2}(|x_io|+|y_io|+|z_io|+|w_io|)} e^{\frac{1}{2}(|x_iz_i|+|y_iw_i|)} \\ &\leq e^{-\frac{1}{2}(|x_io|+|y_io|+|z_io|+|w_io|)} (e^{\frac{1}{2}(|x_iy_i|+|z_iw_i|)} \\ &\quad + e^{\frac{1}{2}(|y_iz_i|+|x_iw_i|)} + \delta e^{\frac{1}{2}\rho_i}) \\ &= e^{-(x_i|y_i)_o-(z_i|w_i)_o} + e^{-(y_i|z_i)_o-(x_i|w_i)_o} + E_i, \end{aligned}$$

where $\rho_i = \max\{|x_i y_i|, |x_x z_i|, |x_i w_i|, |y_i z_i|, |y_i w_i|, |z_i w_i|\}$ and

$$E_i = \delta e^{\frac{1}{2}(\rho_i - |x_i o| - |y_i o| - |z_i o| - |w_i o|)}.$$

Again we see that $E_i \to 0$ and we obtain in the limit the desired ptolemaic inequality.

$\square$

Remark 72. *The above result implies in particular that the asymptotic upper curvature bound (see [BF]) of an asymptotic* PT_κ *space is bounded above by* κ*.*

Chapter 7

Convexities of PT_{κ} Spaces

7.1 More about PT_{κ} spaces

In this section we collect the most important basic facts about geodesic PT_{κ} spaces.

Here we collect a couple of basic properties of PT_{κ} spaces which will be frequently used in the remainder of this chapter.
(P1): Every subset $Y \subset X$ of a Ptolemy metric space X, endowed with the metric inherited from X, is Ptolemy.
(P2): A metric space X is PT_{κ} then for every $\lambda > 0$ the scaled space λX is $\mathrm{PT}_{\frac{\kappa}{\sqrt{\lambda}}}$.
Some of our arguments below will use the notions of ultrafilters and ultralimits; a generalization of pointed Gromov-Hausdorff convergence.The symbol $\lim_w(X_n, x_n)$ will denote such an ultralimit (w.r.t. a non-principal ultrafilter w).

As every metric property, the PT_{κ} condition is invariant w.r.t. ultra- convergence.
(P3): For every sequence $\{(X_i, x_i)\}$ of pointed PT_{κ} spaces and every non-principle ultrafilter w, the ultralimit $\lim_w(X_i, x_i)$ is a PT_{κ} space.

Finally, we recall another important observation, which is due to Schoenberg (see [Sch]).
(P4): A normed vector space is an inner product space if and only if it is Ptolemy.

A subset of a normed vector space is called linearly convex, if with any two points it contains the straight line segment connecting these points. A metric space is called linearly convex, if it is isometric to a linearly convex subset of a normed vector space and called flat, if it is isometric to a convex subset of an inner product space.

With this notation the properties above immediately yield the

Corollary 73. *Let X be a Ptolemy space, then every linearly convex subset $C \subset X$ of X is flat.*

7.2 κ-Busemann convexity

We define next generalized versions of the Busemann convexity. Consider a geodesic triangle of side lengths a, b, c and let m be the length of a segment joining midpoints of the sides of lengths a and b respectively.

Definition 74. *Let X be a geodesic space with diam$(X) < \frac{\pi}{\sqrt{\kappa}}$ for $\kappa > 0$. We say that X is κ-Busemann convex if*

$$\cos(m\sqrt{\kappa}) \geq \frac{1 + \cos(a\sqrt{\kappa}) + \cos(b\sqrt{\kappa}) + \cos(c\sqrt{\kappa})}{4\cos(\frac{a\sqrt{\kappa}}{2})\cos(\frac{b\sqrt{\kappa}}{2})}, \quad \kappa > 0$$

and

$$\cosh(m\sqrt{-\kappa}) \leq \frac{1 + \cosh(a\sqrt{-\kappa}) + \cosh(b\sqrt{-\kappa}) + \cosh(c\sqrt{-\kappa})}{4\cosh(\frac{a\sqrt{-\kappa}}{2})\cosh(\frac{b\sqrt{-\kappa}}{2})}, \quad \kappa < 0$$

where a, b, c, m as above.

Notice that one could define the 0-Busemann convexity as the Busemann convexity in the classical sense. From the definition, it is clear that a CAT(κ) space (of diameter $< \frac{\pi}{\sqrt{\kappa}}$ for $\kappa > 0$) is κ-Busemann convex.

Remark 75. *Let X be a geodesic space that is κ-Busemann convex for $\kappa < 0$. Then X is Busemann convex.*

7.3 Blowing up of PT_κ and κ-Busemann convex spaces

Let X be a geodesic metric space which satisfies PT_κ and κ-Busemann convex conditions and let γ be a geodesic in X emanating from $p \in X$. Now take a non-principle ultrafilter w, consider the w-blow up (X, d) of X in p, i.e. $(X, d) := \lim_w\{(nX, p)\}_n$, and define $\bar{\gamma} : [0, \infty) \to X$ through $\gamma(s) := \lim_w\{(\gamma(\frac{s}{n}), p)\}_n$ for all $s \in [0, \infty)$. This map indeed is a geodesic ray in (X, d) emanating in $\{p\}_n \in \bar{X}$. We call $\bar{\gamma}$ the ultraray associated to γ (and w).

Here is an observation of $\bar{X}$:

Proposition 76. *$\bar{X}$ is a PT_0 geodesic space and often convex.*

Since often convex spaces admit continuous midpoint maps combining with the following Theorem, we immediate obtain the next consequence.

Theorem 77. *[FLS] Let X be a geodesic, Ptolemy space which admits a continuous midpoint map. Then X is uniquely geodesic.*

Corollary 78. *$\bar{X}$ is a PT_0 geodesic space and Busemann convex.*

7.4 Weak angles

In order to get a grip on the interplay between geodesics and their associated ultrarays, we recall certain notions of angles.

Given three points p, x and y in a metric space X, consider corresponding comparison points p', x' and y' in the Euclidean plane $\mathbb{E}^2$. Let $[p', x']$ and $[p', y']$ denote the geodesic segments in $\mathbb{E}^2$ connecting p' to x' and p' to y'. These segments enclose an angle in p' and this angle is referred to as the (Euclidean) comparison angle of x and y at p. We write $\angle_p(x, y)$ for this angle.

Let now X be a metric space and consider two geodesic segments γ_1 and γ_2 parameterized by arclength, both initiating in some $p \in X$. Then γ_1 and γ_2 are said to enclose the angle $\angle_p(\gamma_1, \gamma_2)$ (in the strict sense) at p if the limit $\angle_p(\gamma_1, \gamma_2) := \lim_{s,t\to 0} \angle_p(\gamma_1(s), \gamma_2(t))$ exists.

Recall that for instance a normed vector space is an inner product space if and only if all straight line segments emanating from the origin enclose an angle. However, even in normed vector spaces that are not inner product spaces certain so called generalized angles do exist between any straight line segments initiating in a common point.

Let $a, b > 0$ and γ_1 and γ_2 be as above. then we say that γ_1 and γ_2 enclose a generalized angle $\angle_g(\gamma_1, a, \gamma_2, b)$ at scale (a, b), if the limit

$$\angle_g^p(\gamma_1, a, \gamma_2, b) := \lim_{s\to 0} \angle_p(\gamma_1(as), \gamma_2(bs))$$

exists. If γ_1 and γ_2 enclose generalized angles at all scales (a, b) and, moreover, these generalized angles do not depend on the particular scale, then we say that γ_1 and γ_2 enclose the weak angle

$$\angle_p^w(\gamma_1, \gamma_2) := \angle_p^g(\gamma_1, 1, \gamma_2, 1).$$

The following Lemma is a important observation for κ-Busemann convex spaces

Lemma 79. *[Oh] Let X be κ-Busemann convex and γ_1 and γ_2 be geodesics on X with $\gamma_1(0) = p = \gamma_2(0)$. Then, for all scales (a, b), the limit*

$$\lim_{\epsilon\to 0} \frac{|\gamma_1(a\epsilon)\gamma_2(b\epsilon)|}{\epsilon}$$

exists. In particular, the generalized angle $\angle_p^g(\gamma_1, a, \gamma_2, b)$ exists.

Next consider the ultrarays $\overline{\gamma_1}$ and $\overline{\gamma_2}$ associated to γ_1 and γ_2 . These ultrarays satisfy

$$\bar{d}(\overline{\gamma_1}(as), \overline{\gamma_2}(bs)) = s\bar{d}(\overline{\gamma_1}(a), \overline{\gamma_2}(b)) \quad \forall a, b, s > 0. \tag{7.1}$$

Moreover, the existence of weak angles of geodesics γ_1 and γ_2 in a κ-Busemann convex space is equivalent to the existence of angles (in the strict sense) between their associated ultrarays $\bar{\gamma_1}$ and $\bar{\gamma_2}$ in $\bar{X}$.

Lemma 80. *Let X be κ-Busemann convex, let γ_1 and γ_2 denote geodesics in X initiating in a common point $p \in X$ and let γ_1 and γ_2 denote their associated ultrarays. Then the following properties are mutually equivalent.*

1. *The rays γ_1 and γ_2 enclose a weak angle.*
2. *The ultrarays $\overline{\gamma_1}$ and $\overline{\gamma_2}$ enclose an angle (in the strict sense).*
3. *The union $\gamma_1(\mathbb{R}^+)$ and $\gamma_2(\mathbb{R}^+)$ admits an isometric embedding into the Euclidean plane $\mathbb{E}^2$.*

Proof. The equivalence of (2) and (3) follows immediately from Equation 7.1. Moreover, this equation also implies that the ultrarays $\overline{\gamma_1}$ and $\overline{\gamma_2}$ enclose an angle if and only if they enclose a weak angle. Hence the equivalence of (1) and (2) is a consequence of $\frac{1}{s}\bar{d}(\overline{\gamma_1}(as), \overline{\gamma_2}(bs)) = \lim_w\{\frac{1}{n}|\gamma_1(a/n)\gamma_2(b/n)|\}_n$, and the fact that the generalized angles between γ_1 and γ_2 exist for all scales in any case. $\square$

Proposition 81. *Let X be κ-Busemann convex. Assume that for all geodesic segments γ_1 and γ_2 with $\gamma_1(0) = p = \gamma_2(0)$, the weak angle $\angle_p^w(\gamma_1, \gamma_2)$ exists. Then X is a $\mathrm{CAT}(\kappa)$ space.*

7.5 A convex hull proposition

Proposition 82. *[FLS] Let X be Busemann convex and let $\gamma_1, \gamma_2 : I \to X$ be two linearly reparameterized (finite or infinite) geodesics in X such that $t \to |\gamma_1(t)\gamma_2(t)|$ is affine. Then the convex hull of γ_1 and γ_2 is a convex subset of a two-dimensional normed vector space.*

Given a geodesic metric space X, a function $f : X \to \mathbb{R}$ is called affine if its restriction to each affinely parameterized geodesic γ in X satisfies $f(\gamma(t)) = at + b$ for some numbers $a, b \in R$ that may depend on γ. We say that affine functions on X separate points, if for each pair of distinct points $x, x' \in X$ there is an affine function $f : X \to \mathbb{R}$ with $f(x) = f(x')$. With this terminology the following theorem has been proven in

Theorem 83. *[HL] Let X be a geodesic metric space. If affine functions on X separate points then X is isometric to a convex subset of a normed vector space with a strictly convex norm.*

We can characterize $\mathrm{CAT}(\kappa)$ spaces using PT_κ inequality and κ-Busemann convexity.

Theorem 84. *A geodesic space is $\mathrm{CAT}(\kappa)$ if and only if it is PT_κ and κ-Busemann convexity.*

For the proof, we just have to slightly change the argument in [FLS].

Proof. Every $\mathrm{CAT}(\kappa)$-space is both, PT_κ and κ- Busemann convex. It remains to show that a PT_κ and κ- Busemann convex metric space is already $\mathrm{CAT}(\kappa)$. In order to reach a contradiction, suppose that X is PT_κ and κ- Busemann convex but not $\mathrm{CAT}(\kappa)$. Then, due to Proposition 81 there do exist two geodesics γ_1 and γ_2 that do not enclose a weak angle at their common starting point $p = \gamma_1(0) = \gamma_2(0)$. Let $\overline{\gamma_1}$ and $\overline{\gamma_2}$ denote the geodesic rays defined in $Y = \lim_w\{nX, p\}_n$ as above. Then, due to Lemma 80, $\overline{\gamma_1}$ and $\overline{\gamma_2}$ do not enclose an angle in $\{p\}_n \in Y$ either. Now Y is Busemann convex by Corollary 78. Moreover, the function $t \to |\overline{\gamma_1}(t)\overline{\gamma_2}(t)|$ is linear. Thus, by Proposition 82, the convex hull C of $\overline{\gamma_1}$ and $\overline{\gamma_2}$ is isometric to a convex set of a two-dimensional normed vector space. Since Y is Ptolemy,C is flat by Corollary 73. It follows that $\overline{\gamma_1}$ and $\overline{\gamma_2}$ enclose an angle, which yields the desired contradiction. $\square$

7.6 Convexities

Here we will give the so called weak busemann convexity. For any geodesic triangle ABC in a open ball $B_p(r)$, $0 < t < 1$, let $X_t \in AB$ and $Y_t \in AC$ such that $d(A, X_t) = td(A, B)$ and $d(A, Y_t) = td(A, C)$, respectively.
Weak busemann convexity: if $d(X_t, Y_t) \leq G(r)td(B, C)$, where $G(r) \to 1$ as $r \to 0$.

Remark 85. *$CAT(k)$ spaces satisfy the weak busemann convexity.*

Theorem 86. *For a locally compact geodesically complete geodesic space (X, d_X) if it satisfies the PT_κ inequality and weak busemann convexity, then $\forall x \in X$ the tangent cone $(C\Sigma_x X, d)$ is a $\mathrm{CAT}(0)$ space.*

To prove the proposition we need the following lemma

Lemma 87. *For a locally compact geodesically complete weak busemann convex metric space and $x \in X$, the space of directions Σ_x is compact.*

Proof. Let $x \in X$ and $B'_r(x)$ be a normal neighborhood of x. Consider a metric sphere $S_r(x)$, $0 < r < r'$. The shortest paths connecting x with points of $S_r(x)$ fill in $B'_r(x)$ and each geodesic starting at x can be extended to $S_r(x)$ and is a shortest path within $B'_r(x)$.

The map associating to each point $y \in S_r(x)$ the direction at x of the unique shortest path $[xy]$ is continuous(because of the weak busemann convexity) and $S_R(x)$ is compact. So the image of $S_r(x)$ which is just Σ_x is compact as well. □

Now we begin to prove the Theorem 86

Proof. We first show that $(C\Sigma_x X, d)$ is geodesic. Fix two points $[\gamma, s], [\xi, t] \in (C\Sigma_x X$, where we denote by $[\gamma, s]$ the equivalent class containing $(\gamma, s) \in \Sigma_x \times [0, \infty)$. For $\epsilon > 0$, set $y_\epsilon := \frac{1}{2}\gamma(s\epsilon) + \frac{1}{2}\xi(t\epsilon) \in X$. Since $\frac{d(x, y_\epsilon)}{\epsilon}$ is bounded, we can find a subsequence ϵ_i such that $\lim_{i\to\infty} \frac{d_X(x, y_{\epsilon_i})}{\epsilon_i} = c$. Let $v_{\epsilon_i} := [\gamma_{xy_{\epsilon_i}}, c] \in C\Sigma_x X$ and recall that y_ϵ is the unique midpoint of $\gamma(s\epsilon)$ and $\xi(t\epsilon)$. By the weak busemann convexity, we have

$$\begin{aligned}
d([\gamma, s], v_{\epsilon_i}) &= \lim_{\delta\to 0} \frac{1}{\epsilon\delta} d_X(\gamma(s\epsilon\delta), \gamma_{xy_{\epsilon_i}}(c\epsilon\delta)) \\
&\leq \frac{1}{\epsilon} G(\epsilon r) d_X(\gamma(s\epsilon), \gamma_{xy_{\epsilon_i}}(c\epsilon)) \\
&= G(\epsilon r) \frac{d_X(\gamma(s\epsilon), \xi(t\epsilon))}{2\epsilon} + o(1) \\
&\to \frac{1}{2} d([\gamma, s], [\xi, t])
\end{aligned}$$

as ϵ_i tends to 0. Thus v_{ϵ_i} are approximate midpoints between $[\gamma, s]$ and $[\xi, t]$. Since Σ_x is compact which means there are some accumulate points of v_{ϵ_i}. Moreover, $C\Sigma_x X$ is complete hence it's geodesic space. By taking a scaling limit, it gets that $(C\Sigma_x X, d)$ is Busemann convex and ptolemy, thus it's CAT(0). □

7.7 Uniformly continuous midpoint and strictly convexity

Definition 88. *Let X be a geodesic space. We say that X admits a uniformly continuous midpoint map if there exists a map $m : X \times X \to X$ such that*

$$d(x, m(x, y)) = d(y, m(x, y)) = \frac{d(x, y)}{2}$$

for all $x, y \in X$, and for $n \in N$ and $x_n, x_n', y_n, y_n' \in X$ with

$$\lim_{n\to\infty} d(x_n, x_n') = 0$$

and

$$\lim_{n\to\infty} d(y_n, y_n') = 0$$

we have that

$$\lim_{n\to\infty} d(m(x_n, y_n), m(x_n', y_n')) = 0.$$

Theorem 89. *A geodesic* PT_0 *space with a uniformly continuous midpoint map is strictly convex.*

Here the prove is quite similar as in [MS1]. For the proof we need the following elementary

Lemma 90. *Let $f : [0, a] \to \mathbb{R}$ be a 1-Lipschitz convex function with $f(0) = 0$. For $t > 0$ define $g : (0, a] \to \mathbb{R}$ such that $f(t) = tg(t)$. Then $g(0) = \lim_{t\to 0} g(t)$ exists and $-1 \le g(0) \le 1$.*

Proof. (of the Theorem) Since we already know that the distance function d_p is convex, it suffices to show that for $x, y \in X$ with $|xy| > ||px| - |py||$ there exits a midpoint $m \in m(x, y)$ such that for $|pm| < \frac{1}{2}(|px| + |py|)$. Using this, it is not hard to see that the midpoint is unique.

We choose a geodesic px from p to x and a geodesic py from p to y. For $t > 0$ small, let $x_t \in px$ and $y_t \in py$ be the points with $|x_t x| = t$ and $|y_t y| = t$. We choose geodesics $x_t y_t$ from x_t to y_t. For fixed t small enough there exists by continuity a point $w_t \in x_t y_t$ with $|xw_t| = |w_t y|$. By triangle inequality $|xw_t| = |w_t y| \ge a := \frac{1}{2}|xy|$. Choose x_t' or y_t' as the following way: if $|x_t w_t| \le |w_t y_t|$, then we take y_t' which lies on the geodesic part $w_t y_t$ satisfing $|x_t w_t| = |w_t y_t'|$ and $x_t = x_t'$. If $|x_t w_t| > |w_t y_t|$, then we take x_t' which lies on the geodesic part $x_t w_t$ satisfing $|x_t' w_t| = |w_t y_t|$ and $y_t = y_t'$. Not difficult to see that

$$\lim_{t\to 0} |x_t x_t'| = 0 \ \text{ and } \ \lim_{t\to 0} |y_t y_t'| = 0$$

Using the unifromly continuous of midpoints in (X, d), it is elementary to show that there exists a sequence $t_i \to 0$, such that $\lim_{i\to\infty} w_{t_i} = m$ and $m \in m(x, y)$. Hence the function $\varphi(t_i) = |w_{t_i} m| \to 0$ as $t_i \to 0$.

Let us assume to the contrary that

$$|pm| = \frac{1}{2}(|px| + |py|)$$

We have $a = \frac{1}{2}|xy| = |xm| = |my|$. Let $b = |px|, c = |pm|, d = |py|$ and we assume w.l.o.g that $b \leq c \leq d$. By assumption we have $2c = b + d$. We write

$$|mx_t| = a + ta_x(t), |my_t| = a + ta_y(t)$$

with the functions $a_x(t), a_y(t)$ according to the Lemma. The PT inequality applied to p, x_{t_i}, m, y_{t_i} gives

1. $(a+t_i a_x(t_i))(d-t_i)+(a+t_i a_y(t_i))(b-t_i) \geq c|x_{t_i} y_{t_i}|$. The sum of the PT_0 inequalities for x, x_{t_i}, w_{t_i}, m and m, w_{t_i}, y_{t_i}, y give that

2. $a(a+t_i a_x(t_i))+a(a+t_i a_y(t_i)) \leq a|x_{t_i} y_{t_i}|+2t_i\varphi(t_i)$. From (1) and (2) we obtain

3. $(a+t_i a_x(t_i))(d-t_i)+(a+t_i a_y(t_i))(b-t_i) \geq c((a+t_i a_x(t_i)+a+t_i a_y(t_i))- 2\frac{c}{a}t_i\varphi(t_i)$. Note that by the assumption $2c = b + d$. Thus

4. $(d-c)a_x(0)+(b-c)a_y(0) \geq 2a$. Since $0 \leq (d-c) \leq a$ and $0 \geq (b-c) \geq -a$ and $-1 \leq a_x(0), a_y(0) \leq 1$ this implies that

5. $a_x(0) = 1, a_y(0) = -1$ and $d - c = a, c - b = a$. Hence $|xy| = ||px| - |py||$ in contradiction to the assumption.

$\square$

Chapter 8

A Flat Strip Theorem for Ptolemaic Spaces

In this Chapter we prove a flat strip theorem for geodesic ptolemaic spaces PT_0. Two unit speed geodesic lines $c_0, c_1 : \mathbb{R} \to X$ are called *parallel*, if their distance is sublinear, i.e. if $\lim_{t\to\infty} \frac{1}{t} d(c_0(t), c_1(t)) = \lim_{t\to-\infty} \frac{1}{t} d(c_0(t), c_1(t)) = 0$.

Theorem 91. *Let X be a geodesic PT_0 space which is homeomorphic to $\mathbb{R} \times [0,1]$, such that the boundary curves are parallel geodesic lines, then X is isometric to a flat strip $\mathbb{R} \times [0,a] \subset \mathbb{R}^2$ with its euclidean metric.*

In this section we collect the most important basic facts about geodesic PT_0 spaces which we will need in our arguments. If we do not provide proofs in this section, these can be found in [FLS], [FS2].

Let X be a metric space. By $|xy|$ we denote the distance between points $x, y \in X$. We will always parametrize geodesics proportionally to arclength. Thus a geodesic in X is a map $c : I \to X$ with $|c(t)c(s)| = \lambda|t-s|$ for all $s, t \in I$ and some constant $\lambda \geq 0$. A metric space is called geodesic if every pair of points can be joined by a geodesic.

In addition we will use the following convention in this paper. If a geodesic is parametrized on $[0,\infty)$ or on $\mathbb{R}$, the parametrization is *always* by arclenth. A geodesic $c : [0,\infty) \to X$ is called a *ray*, a geodesic $c : \mathbb{R} \to X$ is called a *line*.

In the sequel X will always denote a geodesic metric space. For $x, y \in X$ we denote by $m(x,y) = \{z \in X \mid |xz| = |zy| = \frac{1}{2}|xy|\}$ the set of midpoints of x and y. A subset $C \subset X$ is *convex*, if for $x, y \in C$ also $m(x,y) \subset C$.

A function $f : X \to \mathbb{R}$ is *convex* (resp. *affine*), if for all geodesics $c : I \to X$ the map $f \circ c : I \to \mathbb{R}$ is convex (resp. affine).

The space X is called *distance convex* if for all $p \in X$ the distance function $d_p = |\cdot\ p|$ to the point p is convex. It is called *strictly distance convex*, if the functions $t \mapsto (d_p \circ c)(t)$ are strictly convex whenever $c : I \to X$ is a geodesic with $|c(t)\,c(s)| > ||p\,c(t)| - |p\,c(s)||$ for all $s, t \in I$, i.e., neither $c(t)$ and $c(s)$ being on a geodesic from p to the other. This definition is natural, since the restriction of d_p to a geodesic segment containing p is never strictly convex. The Ptolemy property easily implies:

Lemma 92. *A geodesic PT_0 space is distance convex.*

As a consequence, we obtain that for PT_0 metric spaces local geodesics are geodesics. Here we call a map $c : I \to X$ a *local geodesic*, if for all $t \in I$ there exists a neighborhood $t \in I' \subset I$, such that $c_{|I'}$ is a geodesic.

Lemma 93. *[FS2] If X is distance convex, then every local geodesic is globally minimizing.*

In [FLS] we gave examples of PT_0 spaces which are not strictly distance convex. However, if the space is proper, then the situation is completely different.

Theorem 94 ([FS2])**.** *A proper, geodesic PT_0 space is strictly distance convex.*

Since we have a relatively short proof of this result, we present the proof in section 8.2 .

Corollary 95 ([FLS])**.** *Let X be a proper, geodesic PT_0 space. Then for $x, y \in X$ there exists a unique midpoint $m(x, y) \in X$. The midpoint function $m : X \times X \to X$ is continuous.*

Corollary 96 ([FS2])**.** *Let X be a proper, geodesic PT_0 space, and $A \subset X$ be a closed and convex subset. Then there exists a continuous projection $\pi_A : X \to A$.*

Remark 97. *For $\mathrm{CAT}(0)$ spaces this projection is always 1-Lipschitz. We do not know if π_A is 1-Lipschitz for general proper geodesic PT_0 spaces.*

The strict convexity of the distance function together with the properness implies easily (cf. Corollary 95)

Corollary 98. *Let X be a proper, geodesic PT_0 space and let $x, y \in X$. Then there exists a unique geodesic $c_{xy} : [0, 1] \to X$ from x to y and the map $X \times X \times [0, 1] \to X$, $(x, y, t) \mapsto c_{xy}(t)$ is continuous.*

We call two rays $c, c' : [0, \infty) \to X$ *asymptotic*, if $\lim_{t\to\infty} \frac{1}{t}|c(t)c'(t)| = 0$.

Corollary 99. *Let X be a proper geodesic PT_0 space and $c_1, c_2 : [0, \infty) \to X$ asymptotic rays with the same initial point $c_1(0) = c_2(0) = p$. Then $c_1{=}c_2$.*

Proof. Assume that there exists $t_0 > 0$ such that $x = c_1(t_0) \neq c_2(t_0) = y$. Let $m = m(x, y)$. By Theorem 94 we have $|pm| < t_0$. Let $\delta = t_0 - |pm| > 0$. For $t > t_0$ consider the points $x, y, x_t = c_1(t_0 + t), y_t = c_2(t_0 + t)$. Note that $\frac{1}{t}|x_t y_t| \to 0$ by assumption. We write $|xy_t| = t{+}\alpha_t$ with $0 \leq \alpha_t$ and $|yx_t| = t{+}\beta_t$ with $0 \leq \beta_t$. The PT_0 inequality applied to the four points gives

$$(t + \alpha_t)(t + \beta_t) \leq t^2 + |xy|\,|x_t y_t|$$

and thus $(\alpha_t{+}\beta_t) \leq \frac{1}{t}|x_t y_t|\,|xy| \to 0$. Thus for t large enough $\alpha_t \leq \delta$. Therefore $|y_t m| \leq \frac{1}{2}(|y_t x|{+}|y_t y|) \leq t{+}\delta/2$, which gives the contradiction $(t{+}t_0) = |py_t| \leq |pm| + |my_t| \leq (t + t_0 - \delta/2)$. $\square$

We now collect some results on the Busemann functions of asymptotic rays and parallel line.

X denotes always a geodesic PT_0 space. Let $c : [0, \infty) \to X$ be a geodesic ray. As usual we define the *Busemann function* $b_c(x) = \lim_{t\to\infty}(|xc(t)| - t)$. Since b_c is the limit of the convex functions $d_{c(t)} - t$, it is convex.

The following proposition implies that, in a PT_0 space, asymptotic rays define (up to a constant) the same Busemann functions.

Proposition 100 ([FS2]). *Let X be a PT_0 space, let $c_1, c_2 : [0,\infty) \to X$ be asymptotic rays with Busemann functions $b_i := b_{c_i}$. Then $(b_1 - b_2)$ is constant.*

Let now $c : \mathbb{R} \to X$ be a geodesic line parameterized by arclength. Let $c^{\pm} : [0,\infty) \to X$ be the rays $c^+(t) = c(t)$ and $c^-(t) = c(-t)$. Let further $b^{\pm} := b_{c^{\pm}}$.

Lemma 101 ([FS2]). *$(b^+ + b^-) \geq 0$ and $(b^+ + b^-) = 0$ on the line c.*

We now consider Busemann functions for parallel lines.

Proposition 102 ([FS2]). *Let $c_1, c_2 : \mathbb{R} \to X$ be parallel lines with with Busemann functions $b_1^{\pm}$ and $b_2^{\pm}$. Then $(b_1^+ + b_1^-) = (b_2^+ + b_2^-)$.*

Corollary 103. *If $c_1, c_2 : \mathbb{R} \to X$ are parallel lines. Then there are reparametrizations of c_1, c_2 such that $b_1^+ = b_2^+$ and $b_1^- = b_2^-$.*

Proof. Since $b_1^+ - b_2^+$ is constant by Proposition 100 we can obviously shift the parametrization of c_2 such that $b_1^+ = b_2^+$. It follows now from Proposition 102 that then also $b_1^- = b_2^-$. □

Corollary 104. *Let X be a geodesic space which is covered by parallels to a line $c : \mathbb{R} \to X$; i.e. for any point $x \in X$ there exists a line c_x parallel to c with $x = c_x(0)$. Then the Busemann functions $b^{\pm}$ of c are affine.*

Proof. We show that $b^+ + b^- = 0$. Let therefore $x \in X$ and let $b_x^{\pm}$ be the Busemann functions of c_x. By Proposition 102 $b^+ + b^- = b_x^+ + b_x^-$. Now $(b_x^+ + b_x^-)(x) = 0$, hence $(b^+ + b^-)(x) = 0$. Thus the sum of the two convex functions b^+ and b^- is affine. It follows that b^+ and b^- are affine. □

More generally the following holds:

Corollary 105. *Let $c : \mathbb{R} \to X$ be a line, then the Busemann functions $b^{\pm}$ are affine on the convex hull of the union of all lines parallel to c.*

Proof. Indeed the above argument shows that $b^+ + b^-$ is equal to 0 on all parallel lines. Since $b^+ + b^-$ is convex and ≥ 0 by Lemma 101, $b^+ + b^- = 0$ on the convex hull of all parallel lines. Thus b^+ and b^- are affine on this convex hull. □

8.1 Proof of the Main Result

We prove a slightly stronger version of the main Theorem, namely:

Theorem 106. *Let X be a geodesic PT_0 space which is topologically a connected 2-dimensional manifold with boundary ∂X, such that the the boundary consists of two parallel geodesic lines. Then X is isometric to a flat strip $\mathbb{R} \times [0,a] \subset \mathbb{R}^2$ with its euclidean metric.*

Using Corollary 103 we can assume that $\partial X = c(\mathbb{R}) \cup c'(\mathbb{R})$, where $c, c' : \mathbb{R} \to X$ are parallel lines with the same Busemann functions $b^{\pm}$. In particular $b^+(c(t)) = b^+(c'(t)) = -t$ and $b^-(c(t)) = b^-(c'(t)) = t$. Let $a := |c(0)c'(0)|$ and for $t \in \mathbb{R}$ let $h_t : [0,a] \to X$ the geodesic from $c(t)$ to $c'(t)$. We emphasize here, that h_0 is parametrized by arclength, but we do not know, if h_t has unit speed for $t \neq 0$. We also define $c_0 := c$ and $c_a := c'$. Define $h : \mathbb{R} \times [0,a] \to X$

by $h(t,s) = h_t(s)$. With H_t we denote the set $h_t([0,a])$. By Corollary 105 the Busemann functions $b^{\pm}$ are affine on the image of h and thus $b^+(h(t,s)) = -t$ and $b^-(h(t,s)) = t$ on H_t.

We claim that h is a homeomorphism: Clearly h is continuous by Corollary 98. To show injectivity we note first that $H_t \cap H_{t'} = \emptyset$ for $t \neq t'$ since b^+ has different values on the sets and secondly that for fixed t the map h_t is clearly injective. Since c_0, c_a are parallel, i.e. the length of h_t is sublinear, we easily see that h is a proper map. Since ∂X is in the image of h and $\mathbb{R} \times (0,a)$, $X \setminus \partial X$ are 2-dimensional connected manifolds and h is injective and proper, we see that h is a homeomorphism.

Lemma 107. *For all $x \in X$ there exists a unique line $c_x : \mathbb{R} \to X$ with c_x parallel to c_0 and c_a and $c_x(0) = x$.*

Proof. The Uniqueness follows from Corollary 99. To show the existence let $x \in H_{t_0}$. Consider for i large enough the unit speed geodesics $c_i^+ : [0, d_i] \to X$ from x to $c_0(i)$, where $d_i = |xc_0(i)|$. By local compactness a subsequence will converge to a limit ray $c_x^+ : [0,\infty) \to X$ with $c_x^+(0) = x$. For topological reasons c_x^+ intersects H_t for $t \geq t_0$. Let $c_x^+(\varphi(t)) \in H_t$, then the sublinearity of the length of H_t implies that $\varphi(t)/t \to 1$ and that c_x^+ is asymptotic to c_0. Furthermore the convex function b^+ has slope -1 on c_x^+, i.e. $b^+(c_x^+(t)) = -t_0 - t$.

In a similar way we obtain a ray $c_x^- : [0,\infty) \to X$ with $c_x^-(0) = x$, c_x^- asymptotic to c_0^- with $b^+(c_x^-(t)) = -t_0 + t$. Now define $c_x : \mathbb{R} \to X$ by $c_x(t) = c_x^+(t)$ for $t \geq 0$ and $c_x(t) = c_x^-(-t)$ for $t \leq 0$. Then c_x is a line since

$$2t \geq |c_x(t)c_x(-t)| \geq |b^+(c_x(t)) - b^+(c_x(-t))| = 2t,$$

and hence $|c_x(t)c_x(-t)| = 2t$. $\square$

For $s \in [0,a]$ let $c_s := c_{h(0,s)}$ be the parallel line through $h(0,s)$. Consider $c : \mathbb{R} \times [0,a] \to X$, $c(t,s) = c_s(t)$. This is another parametrization of X. Note that $b^+(c_s(t)) = -t$.

Remark 108. *We do not know at the moment whether $c(t,s) = h(t,s)$, our final result will imply that.*

Since we have the foliation of X by the lines c_s, we have the property:

(A): If $t, t' \in \mathbb{R}$, $x \in H_t$, then there exist $x' \in H_{t'}$ with $|xx'| = |t - t'|$.

For $0 \leq s \leq a$ we define the *fibre distance* $A_s : X \to \mathbb{R}$ in the following way. Let $x \in X$, $x = c_{s'}(t')$, i.e. $x \in H_{t'}$. Then $A_s(x) = \pm|xc_s(t')|$, where the sign equals the sign of $(s' - s)$. Thus $A_s(x)$ is the distance in the fibre $H_{t'}$ from the point x to the intersection point $c_s(\mathbb{R}) \cap H_{t'}$. Note that by easy triangle inequality arguments A_s is a 2-Lipschitz function.

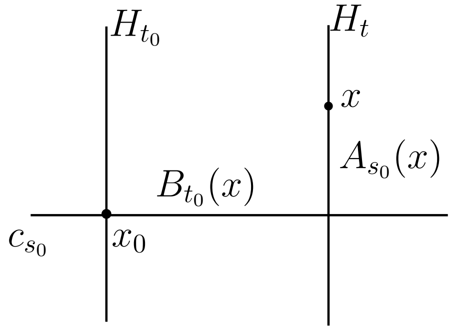
We also define for $t \in \mathbb{R}$ the function $B_t : X \to \mathbb{R}$ by $B_t(x) = (t' - t)$, when $x \in H_{t'}$. Note that B_t is 1-Lipschitz and affine, since b^+ is 1-Lipschitz and affine.

For fixed $x_0 = c_{s_0}(t_0) \in X \setminus \partial X$ consider the map $F_{x_o} : X \to \mathbb{R}^2$ defined by

$$F_{x_0}(x) = (B_{t_0}(x), A_{s_0}(x)).$$

Lemma 109. *F_{x_0} is a bilipschitz map, where $\mathbb{R}^2$ carries the standard euclidean metric d_{eu}, more precisely for all $x, y \in X$ we have*

$$\frac{1}{4}|xy| \leq d_{\mathrm{eu}}(F_{x_0}(x), F_{x_0}(y)) \leq 2|xy|.$$



Proof. Since B_{t_0} is 1-Lipschitz and A_{s_0} is 2-Lipschitz, also F_{x_0} is 2-Lipschitz. Now assume $x \in H_t$, $y \in H_{t'}$. We claim that $|F_{x_0}(x) - F_{x_0}(y)| \geq \frac{1}{4}|xy|$. To prove this claim, we can assume that $|B_{t_0}(x) - B_{t_0}(y)| \leq \frac{1}{4}|xy|$. By Property (A) there exists $x' \in H_{t'}$ with $|xx'| = |t - t'| \leq \frac{1}{4}|xy|$. Thus $|x'y| \geq \frac{3}{4}|xy|$ and hence

$$|A_{s_0}(y) - A_{s_0}(x)| \geq |A_{s_o}(y) - A_{s_0}(x')| - |A_{s_0}(x') - A_{s_0}(x)|.$$

Note that

$$|A_{s_o}(y) - A_{s_0}(x')| = |yx'|$$

and

$$|A_{s_0}(x') - A_{s_0}(x)| \leq 2|x'x|,$$

since A_{s_0} is 2-Lipschitz. Thus

$$|A_{s_0}(y) - A_{s_0}(x)| \geq |yx'| - 2|x'x| \geq \frac{1}{4}|xy|.$$

□

For $\lambda > 0$ we define $F^\lambda_{x_0} : X \to \mathbb{R}^2$ by $F^\lambda_{x_0}(x) = \lambda F_{x_0}(x)$. Then $F^\lambda_{x_0} : (X, \lambda d) \to (\mathbb{R}^2, d_{\text{eu}})$ is also a bilischitz with the same constants $\frac{1}{4}$ and 2 for all $\lambda > 0$. Now consider an increasing sequence $\lambda_i \to \infty$ and let d_{λ_i} be the metric on $W_{\lambda_i} = \lambda_i \cdot (F_{x_0}(X)) \subset \mathbb{R}^2$ such that $F^{\lambda_i}_{x_0} : (X, \lambda_i d) \to (W_i, d_{\lambda_i})$ is an isometry. By the above we have $\frac{1}{2}d_{\text{eu}} \leq d_{\lambda_i} \leq 4d_{\text{eu}}$.

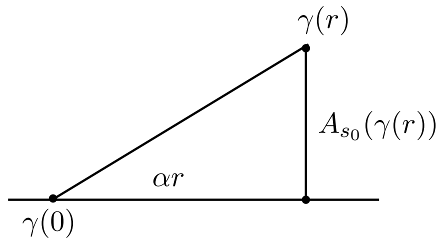
Proposition 110. *If $\lambda_i \to \infty$ then d_{λ_i} converges uniformly on compact subsets to the standard euclidean distance d_{eu}.*

Proof. Since x_0 is an inner point of X, $W_{\lambda_1} \subset W_{\lambda_2} \subset \cdots$ and $\bigcup W_{\lambda_i} = \mathbb{R}^2$. Since $\frac{1}{2}d_{\text{eu}} \leq d_{\lambda_i} \leq 4d_{\text{eu}}$ any subsequence of the integers has itself a subsequence $i_j \to \infty$ with $d_{\lambda_{i_j}} \to d_\omega$ for some accumulation metric d_ω on $\mathbb{R}^2$. We show that $d_\omega = d_{\text{eu}}$ is always the the euclidean distance and hence d_{λ_i} will converge to d_{eu}.

To prove this we collect some properties of the accumulation metric d_ω:

(a) $(\mathbb{R}^2, d_\omega)$ is a geodesic PT space.

(b) By construction $F^\lambda_{x_0}$ maps the geodesic c_{s_0} to the line $t \mapsto (t, 0)$ in $\mathbb{R}^2$ and the geodesic segment H_t to a part of the line $s \mapsto (\lambda(t - t_0), s)$. Therefore $t \mapsto (t, 0)$ is a geodesic parametrized by arclength in the metric $(\mathbb{R}^2, d_\omega)$ and



$s \mapsto (t,s)$ is a geodesic parametrized by arclength for all s. Each of these vertical geodesics $s \mapsto (t,s)$ is contained in a level set of the Busemann function b_1 of the line $t \mapsto (t,0)$. Thus $b_1(t,s) = -t$ and b_1 is affine as a limit of affine functions.

(c) The property (A) implies in the limit that for $x = (t,s)$ and $t' \in \mathbb{R}$ there exists $y = (t', s')$ with $|xy| = |t - t'|$. In particular the lines $s \mapsto (t,s)$ are all parallel. Thus if b_2 is the Busemann function of $s \mapsto (0,s)$, then this function is affine by Corollary 104. Note that $b_2(0,s) = -s$ and $b_2(t,s) = b_2(t,0) - s$. Since b_2 is affine and $t \mapsto (t,0)$ is a geodesic, we have $b_2(t,0) = \alpha t$ for some $\alpha \in \mathbb{R}$ and hence $b_2(t,s) = \alpha t - s$.

Thus the two affine functions b_1 and b_2 separate the points in $(\mathbb{R}^2, d_\omega)$. It follows by the result of Hitzelberger-Lytchak [HL], that $(\mathbb{R}^2, d_\omega)$ is isometric to a normed vector space. It follows then from the theorem of Schoenberg [Sch], that $(\mathbb{R}^2, d_\omega)$ is isometric to an inner product space. We claim that the constant α equals 0: Since the line $s \mapsto (0,s)$ lies in some level set of the Busemannn function of the line $t \mapsto (t,0)$ and the space is an inner product space, the two lines are orthogonal, i.e. $\alpha = 0$. It now follows easily that $d_\omega = d_{\mathrm{eu}}$. $\square$

Consider now a unit speed geodesic $\gamma : [0,d] \to X$ with $\gamma(0) = c_{s_0}(t_0) \in X \setminus \partial X$. Since B_{t_0} is affine, we have $B_{t_0}(\gamma(r)) = \alpha r$ for some $\alpha \in \mathbb{R}$.

Corollary 111. *With this notation we have*

$$\lim_{r \to 0} \frac{A_{s_0}^2(\gamma(r))}{r^2} = 1 - \alpha^2.$$

Proof. Note that $F_{x_0}(\gamma(r)) = (\alpha r, A_{s_0}(\gamma(r)))$. By Propossition 110

$$d_{\mathrm{eu}}(0, F_{x_0}^{1/r}(\gamma(r))) \to \frac{1}{r}|x_0\gamma(r)| = 1.$$

Now

$$d_{\mathrm{eu}}^2(0, F_{x_0}^{1/r}(\gamma(r))) = \alpha^2 + \frac{A_{s_0}^2(\gamma(r))}{r^2}.$$

$\square$

Let $\sigma = c_s(\mathbb{R})$ be one of the parallel lines with $0 \le s \le a$ considered as closed convex subset of X. We then have the projection $\pi_\sigma : X \to \sigma$ from Corollary 96. We show that the projection stays in the same fibre.

Lemma 112. $b^+(\pi_\sigma(x)) = b^+(x)$

Proof. It suffices to prove the result for $\sigma = c_s(\mathbb{R})$, where $0 < s < a$, since for $s = 0, a$ it then follows by continuity.

Assume that $\pi_\sigma(x) = x_0 \in H_{t_0}$, while $x \in H_t$. Let $\gamma : [0, d] \to X$ be the unit speed geodesic from x_0 to x where $d = |x_0 x|$. Let $D : X \to [0, \infty)$ be the distance to σ, i.e. $D(x) = |x\pi_\sigma(x)|$. Note that $D(x) \le |A_s(x)|$ and that $D(\gamma(r)) = r$. Since b^+ is affine we have $b^+(\gamma(r)) = \alpha r - t_0$ for some $\alpha \in \mathbb{R}$.

We have to show that $\alpha = 0$. By Corollary 111

$$\lim_{r \to 0} \frac{A_s^2(\gamma(r))}{r^2} = 1 - \alpha^2.$$

If $|\alpha| \ne 0$ this would imply that for $r > 0$ small enough $|A_s(\gamma(r))| < r = D(\gamma(r))$, in contradiction to $D(x) \le |A_s(x)|$.

$\square$

Lemma 113. *For $s_1, s_2 \in [0, 1]$ the function $t \mapsto |c_{s_1}(t)c_{s_2}(t)|$ is constant.*

Proof. Let $c = c_{s_1}$ and $c' = c_{s_2}$.

We put $\mu(t) = |c(t)c'(t)|$. By Lemma 112 $c'(t)$ is a closest to $c(t)$ point on $c'(\mathbb{R})$, and vice versa, $c(t)$ is a closest to $c'(t)$ point on $c(\mathbb{R})$ for every $t \in \mathbb{R}$. Thus $|c(t)c'(t')|$, $|c'(t)c(t')| \ge \max\{\mu(t), \mu(t')\}$ for each t, $t' \in \mathbb{R}$. Applying the Ptolemy inequality to the quadruple $(c(t), c(t'), c'(t'), c'(t))$, we obtain

$$\max\{\mu(t), \mu(t')\}^2 \le |c(t)c'(t')||c'(t)c(t')| \le \mu(t)\mu(t') + (t - t')^2.$$

We show that $\mu(a) = \mu(0)$ for every $a \in \mathbb{R}$. Assume W.L.G. that $a > 0$ and put $m = 1/\min_{0 \le s \le a} \mu(s)$. Then $|\mu(t) - \mu(t')| \le m(t - t')^2$ for each $0 \le t, t' \le a$. Now

$$\mu(a) - \mu(0) = \mu(s) - \mu(0) + \mu(2s) - \mu(s) + \cdots + \mu(a) - \mu((k-1)s),$$

where $s = a/k$ for $k \in \mathbb{N}$. It follows $|\mu(a) - \mu(0)| \le mks^2 = ma^2/k \to 0$ as $k \to \infty$. Hence, $\mu(a) = \mu(0)$.

$\square$

As a consequence we have $|c(t, s)c(t, s')| = |s - s'|$ for all $t \in \mathbb{R}$ and of course we also have $|c(t, s)c(t', s)| = |t - t'|$ for all $s \in \mathbb{R}$. Note that Lemma 113 also implies the formula $A_{s_0}(c(t, s)) = s - s_0$.

We finally want to show that $|c(t, s)c(t', s')| = \sqrt{|t - t'|^2 + |s - s'|^2}$. We assume for simplicity $t' \ge t$ and $s' \ge s$. Let $\gamma : [0, d] \to X$ be a unit speed geodesic from $c(t, s)$ to $c(t', s')$ with $d = |c(t, s)c(t', s')|$. We can write $\gamma(r) = c(\gamma_1(r), \gamma_2(r))$. By our assumption γ_1 and γ_2 are nondecreasing. Since $\gamma_1(r) = B_{t_0}(\gamma(r))$ is affine we have

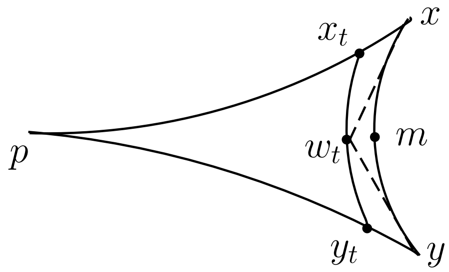
$$\gamma_1(r) = t + \frac{t' - t}{d} r.$$

Note that by the above formula for A_s we have for $r_0, r_1 \in [0, d]$

$$A_{\gamma_2(r_0)}(\gamma(r_1)) = \gamma_2(r_1) - \gamma_2(r_0).$$

Therefore it follows from Corollary 111 that for every $r_0 \in [0, d)$

$$\lim_{r \to 0} \frac{(\gamma_2(r_0 + r) - \gamma_2(r_0))^2}{r^2} = 1 - \frac{(t' - t)^2}{d^2}.$$



This implies that γ_2 is differentiable with derivative

$$\gamma_2'(r_0) = \sqrt{1 - \frac{(t'-t)^2}{d^2}},$$

in particular the derivative is constant and therefore also γ_2 is affine and hence

$$\gamma_2(r) = s + \frac{s'-s}{d} r.$$

The formula for the derivative also implies

$$\frac{s'-s}{d} = \gamma_2'(r_0) = \sqrt{1 - \frac{(t'-t)^2}{d^2}}$$

which finally shows our claim $d^2 = (t'-t)^2 + (s'-s)^2$.

8.2 A short proof of strict convexity

In this section we give a short proof of Theorem 94.

Theorem 114. *A proper, geodesic* PT_0 *metric space is strictly distance convex.*

For the proof we need the following elementary

Lemma 115. *Let* $f : [0,a] \to \mathbb{R}$ *be a* 1*-Lipschitz convex function with* $f(0) = 0$*. For* $t > 0$ *define* $g : (0,a] \to \mathbb{R}$ *such that* $f(t) = tg(t)$*. Then* $g(0) = \lim_{t\to 0} g(t)$ *exists and* $-1 \le g(0) \le 1$*.*

Proof. (of the Theorem) Since we already know that the distance function d_p is convex, it suffices to show that for $x, y \in X$ with $|xy| > ||px| - |py||$ there exits a midpoint $m \in m(x,y)$ such that for $|pm| < \frac{1}{2}(|px| + |py|)$. Using this, it is not hard to see that the midpoint is unique.

We choose a geodesic px from p to x and a geodesic py from p to y. For $t > 0$ small, let $x_t \in px$ and $y_t \in py$ be the points with $|x_t x| = t$ and $|y_t y| = t$. We choose geodesics $x_t y_t$ from x_t to y_t. For fixed t small enough there exists by continuity a point $w_t \in x_t y_t$ with $|x w_t| = |w_t y|$. By triangle inequality $|xw_t| = |w_t y| \ge a := \frac{1}{2}|xy|$. Using the properness of (X,d), it is elementary to show that there exists a sequence $t_i \to 0$, such that $\lim_{i\to\infty} w_{t_i} = m$ and $m \in m(x,y)$. Hence the function $\varphi(t_i) = |w_{t_i} m| \to 0$ as $t_i \to 0$.

Let us assume to the contrary that

$$|pm| = \frac{1}{2}(|px| + |py|)$$

We have $a = \frac{1}{2}|xy| = |xm| = |my|$. Let $b = |px|, c = |pm|, d = |py|$ and we assume w.l.o.g that $b \leq c \leq d$. By assumption we have $2c = b + d$. We write

$$|mx_t| = a + ta_x(t), |my_t| = a + ta_y(t)$$

with the functions $a_x(t), a_y(t)$ according to the Lemma. The PT inequality applied to p, x_{t_i}, m, y_{t_i} gives

1. $(a+t_ia_x(t_i))(d-t_i)+(a+t_ia_y(t_i))(b-t_i) \geq c|x_{t_i}y_{t_i}|$. The sum of the PT inequalities for x, x_{t_i}, w_{t_i}, m and m, w_{t_i}, y_{t_i}, y give that

2. $a(a+t_ia_x(t_i))+a(a+t_ia_y(t_i)) \leq a|x_{t_i}y_{t_i}|+2t_i\varphi(t_i)$. From (1) and (2) we obtain

3. $(a+t_ia_x(t_i))(d-t_i)+(a+t_ia_y(t_i))(b-t_i) \geq c((a+t_ia_x(t_i)+a+t_ia_y(t_i))-2\frac{c}{a}t_i\varphi(t_i)$. Note that by the assumption $2c = b + d$. Thus

4. $(d-c)a_x(0)+(b-c)a_y(0) \geq 2a$. Since $0 \leq (d-c) \leq a$ and $0 \geq (b-c) \geq -a$ and $-1 \leq a_x(0), a_y(0) \leq 1$ this implies that

5. $a_x(0) = 1, a_y(0) = -1$ and $d - c = a, c - b = a$. Hence $|xy| = ||px| - |py||$ in contradiction to the assumption.

□

8.3 4-Point Curvature Conditions

In this section we briefly discuss question (**Q**) stated in the introduction. We discuss it in the context of conditions for the distance between four points in a given metric space. We use the following notation. Let M^4 be the set of isometry classes of 4-point metric spaces. For a given metric space X let $M^4(X)$ the set of isometry classes of four point subspaces of X. We consider three inequalities between the distances of four points x, y, z, w.

The Ptolemaic inequality

$$|xy|\,|zw| \;\leq\; |xz|\,|yw| \;+\; |xw|\,|yz| \tag{8.1}$$

The inequality

$$|xy|^2 \;+\; |zw|^2 \;\leq\; |xz|^2 \;+\; |yw|^2 \;+\; |xw|^2 \;+\; |yz|^2 \tag{8.2}$$

which is called the *quadrilateral inequality* in [BN] and is equivalent to the 2-roundness condition of Enflo [E1].

We also consider the intermediate inequality

$$|xy|^2 \;+\; |zw|^2 \;\leq\; |xz|^2 \;+\; |yw|^2 \;+\; 2\,|xw|\,|yz| \tag{8.3}$$

With [BN] we call it the *cosq* condition. Let us denote with $\mathcal{A}_{PT}, \mathcal{A}_{QI}, \mathcal{A}_{cosq}$ the isometry classes of spaces in M^4, such that for all relabeling of the points

x, y, z, w the conditions (8.1),(8.2),(8.3) hold respectively. Since always $2ab \leq a^2 + b^2$ we clearly have $\mathcal{A}_{cosq} \subset \mathcal{A}_{QI}$, but no other inclusion holds: The space x, y, z, w with $|xy| = 2$ and all other distances equal to 1 shows that $\mathcal{A}_{PT} \neq \mathcal{A}_{QI}$ and the space x, y, z, w with $|xy| = |zw| = 2$, $|xz| = |xw| = 1$ and $|yz| = |yw| = a$ with $1 < a < 2$ and a very close to 2 shows $\mathcal{A}_{cosq} \neq \mathcal{A}_{PT}$.

A CAT(0)-space satisfies all conditions (8.1),(8.2),(8.3), i.e. $M^4(X) \subset \mathcal{A}_{cosq} \cap \mathcal{A}_{PT}$ (see [FLS], [BFW]).

Berg and Nikolaev ([BN], compare also [Sa]) proved a beautiful characterization of CAT(0) spaces:

A geodesic metric space X is CAT(0) if and only if all quadruples in X satisfy the quadrilateral condition (8.2).

This implies also the following characterization:

A geodesic metric space X is CAT(0) if and only if all quadruples in X satisfy the cosq condition (8.3).

Formally speaking [BN] proves: if X is a geodesic metric space with $M^4(X) \subset \mathcal{A}_{QI}$, then X is CAT(0).

The question (Q) asks for a similar characterization in terms of the PT_0 condition. In [FLS] we gave examples of geodesic PT_0 spaces which are not CAT(0). Since these examples are not proper, they leave the question (Q) open. Actually in proper geodesic PT spaces the distance function to a point is strictly convex, see Theorem 94, thus there is some plausibility for a positive answer to the question. Our result is another indication in this direction.

Finally we remark that in [FLS] we characterized CAT(0) spaces by the property that they are geodesic PT spaces which are in addition Busemann convex.

Chapter 9

Menger Curvature and Ptolemy Segments

In this chapter we will introduce the definition of Menger curvature and the relations between ptolemy curves and its menger curvature.

Definition 116. *Let p_0 be a fixed point in a subset G of a metric space X, denote (p_0, G). For each point $q \in G$, there exist a neighborhood in G, $U(q)$ and constants $\lambda(q) \geq 0$ and $0 \leq k(q) < 1$ for which the following condition is satisfied. For any isosceles triple p_1, p_2, p_3 in $U(q)$ with $p_1p_3 = p_2p_3$, the inequality*

$$p_0p_1 \cdot p_2p_3 + p_0p_2 \cdot p_1p_3 - p_0p_3 \cdot p_1p_2 + \lambda(q)(p_1p_2)^{3-3k(q)} \cdot (2p_1p_3 - p_1p_2)^{k(q)} \geq 0$$

holds.

If $G = X$, then X is called weakly ptolemy.

Let $\{x, yz\}$ be a metric triple, $c(x, y, z)$ is defined as

$$c(x, y, z) = \frac{\sqrt{(d_1 + d_2 + d_3)(d_1 + d_2 - d_3)(d_1 - d_2 + d_3)(-d_1 + d_2 + d_3)}}{d_1 d_2 d_3}$$

where $d_1 = d(x, y), d_2 = d(y, z)$ and $d_3 = d(x, z)$.

Karl Menger introduced this definition of Menger curvature. In his terminology a metric space E has at a point p the curvature $K_M(p)$ if $c(x, y, z) \to K_M(p)$ as the distinct points x, y and z converge independently and simultaneously to p.

Theorem 117. *Let γ be a closed curve in a weakly ptolemy metric space X with initial point p_0. If the Menger Curvature along γ, $K(\gamma) \leq M$. Then the length of γ, $l(\gamma) \geq 2diam(\gamma) \geq f(M)$.*

Proof. By the n-lattice theorem, $p_0 = p_0, p_1, \ldots, p_n = b$ exists in γ with $p_ip_j = \alpha(n)$ for $|i - j| = 1$. For brevity, we denote $p_0p_i = d_i$ for $i = 0, 1, \ldots, n$ and $p_{i-1}p_{i+1} = c_i$ for $i = 1, \ldots, n-1$. By the usual compactness argument, for each $\epsilon > 0$ there is a positive integer N such that for $n \geq N$ the curvature of each consecutive triple of points of every n-lattice in γ is less than ϵ. Thus, for $n \geq N$ and $i = 1, \ldots, n-1$ we have

$$K(q_i)-\epsilon < K(p_{i-1}, p_i, p_{i+1}) = \alpha(n)^{-2}(2\alpha(n)+c_i)^{1/2}(2\alpha(n)-c_i)^{1/2} < K(q_i)+\epsilon,$$

which implies

$$\frac{1}{2}(K(q_i) - \epsilon) < \alpha(n)^{-\frac{3}{2}}(2\alpha(n) - c_i)^{1/2} < K(q_i) + \epsilon.$$

Since γ is compact, there exists a finite covering $\{U(q_i)|i = 1, \ldots, m\}$ of γ. Let β be a Lebesgue number of this finite covering and put $\lambda = \max\{\lambda(q_1), \ldots, \lambda(q_m)\}$. Since $n\alpha(n) \leq L$, if $n > L/\beta$ then each consecutive triple of points of every n-lattice in γ will be contained in some $U(q_i)$. Thus for $n > L/\beta$ and $i = 1, \ldots, n-1$,

$$d_{i-1}\alpha(n) + d_{i+1}\alpha(n) - d_i c_i + \lambda\alpha(n)^{3-3k(q_i)}(2\alpha(n) - c_i)^{k(q_i)} \geq 0$$

which can be rewritten in the form

$$d_{i-1} + d_{i+1} - 2d_i + \alpha(n)^{-1}d_i(2\alpha(n) - c_i) + \lambda\alpha(n)^{2-3k(q_i)}(2\alpha(n) - c_i)^{k(q_i)} \geq 0$$

Since $d_i \leq L$, for $n > \max(N, L/\beta)$, the inequalities imply

$$d_{i-1} + d_{i+1} - 2d_i + \alpha(n)^2((M + \epsilon)^2 L + (M + \epsilon)^{2k}\lambda) \geq 0$$

where $k = max(k(q_1), \ldots, k(q_m)) \leq 1$. Then summing from $i = 1$ to $i = n-1$ we obtain

$$p_0 b - n\alpha(n) + \frac{n(n-1)}{2}\alpha(n)^2((M + \epsilon)^2 L + (M + \epsilon)^{2k}\lambda) \geq 0$$

Now $L = \lim n\alpha(n)$ as $n \to \infty$. Therefore, we have

$$p_0 b - L + \frac{L^2}{2}((M + \epsilon)^2 L + (M + \epsilon)^{2k}\lambda) \geq 0$$

Since it holds for all $\epsilon > 0$, we can conclude that

$$L - d \leq \frac{L^2}{2}(M^2 L + M^{2k}\lambda)$$

Now we will prove that L can not be too small. Let $d = \max d(p_0, \gamma(t))$. Suppose that $L - \frac{L^2}{2}(M^2 L + M^{2k}\lambda) > \frac{L}{2}$ (i.e. $d > \frac{L}{2}$), then $L < \frac{\sqrt{M^{4k-2}\lambda^2+4}-M^{2k-1}\lambda}{2M}$ which means

$$\frac{3}{2}L - \frac{27}{16}M^2 L^3 - \frac{9}{8}M^{2k}\lambda L^2 > L$$

. We can obtain that $d > L$ contradiction! Thus

$$L \geq \frac{\sqrt{M^{4k-2}\lambda^2 + 4} - M^{2k-1}\lambda}{2M}$$

□

Definition 118. *An arc γ in a metric space X is called weakly ptolemy circle if we can find a fixed point p_0 in γ, for each point $q \in \gamma$, there exist a neighborhood in γ, $U(q)$ and constants $\lambda(q) \geq 0$ and $0 \leq k(q) \leq 1$ for which the following condition is satisfied. For any isosceles triple p_1, p_2, p_3 in $U(q)$ with $p_1p_3 = p_2p_3$, the inequality*

$$p_0p_1 \cdot p_2p_3 + p_0p_2 \cdot p_1p_3 - p_0p_3 \cdot p_1p_2 - \lambda(q)(p_1p_2)^{3-3k(q)} \cdot (2p_1p_3 - p_1p_2)^{k(q)} \leq 0$$

holds.

Theorem 119. *Let γ be a weakly ptolemy circle segment in a metric space X. If the Menger Curvature along γ, $K(\gamma) \geq M$. Then the length of γ, $l(\gamma) \leq f(M)$.*

Proof. By the n-lattice theorem, $p_0 = p_0, p_1, \ldots, p_n = b$ exists in γ with $p_ip_j = \alpha(n)$ for $|i-j| = 1$. For brevity, we denote $p_0p_i = d_i$ for $i = 0, 1, \ldots, n$ and $p_{i-1}p_{i+1} = c_i$ for $i = 1, \ldots, n-1$. By the usual compactness argument, for each $\epsilon > 0$ there is a positive integer N such that for $n \geq N$ the curvature of each consecutive triple of points of every n-lattice in γ is less than ϵ. Thus, for $n \geq N$ and $i = 1, \ldots, n-1$ we have

$$K(q_i)-\epsilon < K(p_{i-1}, p_i, p_{i+1}) = \alpha(n)^{-2}(2\alpha(n)+c_i)^{1/2}(2\alpha(n)-c_i)^{1/2} < K(q_i)+\epsilon,$$

which implies

$$\frac{1}{2}(K(q_i) - \epsilon) < \alpha(n)^{-\frac{3}{2}}(2\alpha(n) - c_i)^{1/2} < K(q_i) + \epsilon.$$

Since γ is compact, there exists a finite covering $\{U(q_i)|i = 1, \ldots, m\}$ of γ. Let β be a Lebesgue number of this finite covering and put $\lambda = \max\{\lambda(q_1), \ldots, \lambda(q_m)\}$. Since $n\alpha(n) \leq L$, if $n > L/\beta$ then each consecutive triple of points of every n-lattice in γ will be contained in some $U(q_i)$. Thus for $n > L/\beta$ and $i = 1, \ldots, n-1$,

$$d_{i-1}\alpha(n) + d_{i+1}\alpha(n) - d_ic_i - \lambda\alpha(n)^{3-3k(q_i)}(2\alpha(n) - c_i)^{k(q_i)} \leq 0$$

which can be rewritten in the form

$$d_{i-1} + d_{i+1} - 2d_i + \alpha(n)^{-1}d_i(2\alpha(n) - c_i) - \lambda\alpha(n)^{2-3k(q_i)}(2\alpha(n) - c_i)^{k(q_i)} \leq 0$$

Suppose $L > 4k\lambda(m+\epsilon)^{2k-2}$ (i.e. function $\frac{(x-\epsilon)^2}{L} - 2k\lambda(x+\epsilon)^{2k-1}$ increase about x) since $d_i \leq L$, for $n > \max(N, L/\beta)$, the inequalities imply

$$d_{i-1} + d_{i+1} - 2d_i + \alpha(n)^2(\frac{(m-\epsilon)^2}{4}L - (m+\epsilon)^{2k}\lambda) \leq 0$$

where $k = \min(k(q_1), \ldots, k(q_m)) < 1$. Then summing from $i = 1$ to $i = n-1$ we obtain

$$p_0b - n\alpha(n) + \frac{n(n-2)}{2}\alpha(n)^2(\frac{(m-\epsilon)^2}{4}L - (m+\epsilon)^{2k}\lambda) \leq 0$$

Now $L = \lim n\alpha(n)$ as $n \to \infty$. Therefore, we have

$$p_0b - L + \frac{L^2}{2}(\frac{(m-\epsilon)^2}{4}L - (m+\epsilon)^{2k}\lambda) \leq 0$$

Since it holds for all $\epsilon > 0$, we can conclude that

$$L - d \geq \frac{L^2}{2}(\frac{m^2}{4}L - m^{2k}\lambda)$$

we can obtain that

$$L - \frac{L^2}{2}(\frac{m^2}{4}L - m^{2k}\lambda) > 0$$

thus, $L < \frac{2\sqrt{m^{4k-2}\lambda^2+2}+2m^{2k-1}\lambda}{m}$. So

$$L < \max(4k\lambda(m+\epsilon)^{2k-2}, \frac{2\sqrt{m^{4k-2}\lambda^2+2}+2m^{2k-1}\lambda}{m})$$

□

A metric Jordan curve Γ is bounded turning if there is a constant $C \leq 1$ such that for each pair of points $x, y \in \Gamma$, the arc of smaller diameter $\Gamma[x,y] \subset \Gamma$ between x, y satisfies

$$diam\Gamma[x,y] \leq C|x-y|.$$

Lemma 120 (DM11)**.** *A metric Jordan curve γ is bounded turning if and only if there exists a weak-quasisymmetric homeomorhpism $\varphi : S^1 \to \gamma$.*

Lemma 121 (TV80)**.** *A metric Jordan curve is a metric quasicircle if and only if it is both bounded turning and doubling(that is, of finite Assouad dimension).*

Theorem 122. *Suppose a Jordan curve γ is in a weakly ptolemy space X with menger curvature exist and has finite length, then there exists a quasisymmetric homeomorphism $\varphi : S^1 \to \gamma$.*

Proof. We parametritis γ by length, $\gamma : [0, L] \to X$. According to Lemma, we only have to prove that γ is bounded turning and doubling. From Theorem 1, we know that $L - d \leq \frac{L^2}{2}(M^2L + M^{2k}\lambda)$. Thus for any subarc of γ with length $L' \leq c_1L$, c_1 could be very small, then we have $d' > c_2L'$ for some constant c_2. Hence $diam\gamma[x,y] \leq \frac{1}{c_2}d(x,y)$

Since γ is compact, then $\min_{|s-t|\geq c_1L} d(\gamma(s), \gamma(t)) = a > 0$. Now we consider the function

$$f(t) = \frac{1}{a}d(\gamma(t_0), \gamma(t))$$

Since it is continuous at t_0, for $\epsilon = 1$ there exists $\delta > 0$ such that for any two points $\gamma(t_1), \gamma(t_2)$ if $|t_1 - t_2| < \delta$ then $d(\gamma(t_1), \gamma(t_2)) < a$. Choose $\delta' = \min\{\delta, c_1L\}$. we obtain

$$diam\Gamma[\gamma(t), \gamma(s)] < L = \frac{L}{\delta'}\delta' \leq \frac{L}{c_2\delta'}a < \frac{L}{c_2\delta'}d(\gamma(t), \gamma(s))$$

for $|t - s| > c_1L$.
So for $C = \max\{\frac{1}{c_2}, \frac{L}{c_2\delta'}\}$ we have $diam\Gamma[x,y] \leq Cd(x,y)$ for each pair of points $x, y \in \gamma$.

For doubling, since we have

$$d \geq L - \frac{L^2}{2}(M^2L + M^{2k}\lambda)$$

Suppose that

$$\frac{1}{2}(L - \frac{L^2}{2}(M^2 L + M^{2k}\lambda)) \geq \frac{L}{4}$$

Hence, $L \leq c_3$ for some constant c_3. Thus if $L \leq c_3$ we can dived γ into 4 pieces with the same length $\frac{L}{4}$. It's easy to obtain that $diam\gamma_i \leq \frac{1}{2}(diam\gamma), i = 1, 2, 3, 4$.

If $L \geq c_3$, we have to find the minimum $n \in Z$ such that

$$\frac{L}{4^{n_1}} \leq c_3, \frac{c_3}{2^{n_2}} \leq \frac{b}{2}$$

where $b = min_{|t-s| \geq c_3}\{d(\gamma(t), \gamma(s))\}, n = n_1 + n_2$. Choose $N = 4^n = [\frac{4c_3 L}{b^2}] + 1$, we conclude that for every subset in γ of diameter D, there exists a cover that consists of at most N subsets each having diameter at most $\frac{D}{2}$. Hence γ is doubling. □

The same proof shows the following.

Corollary 123. *A metric Jordan arc γ is in a weakly ptolemy space X with menger curvature exist and has finite length, then there exists a quasisymmetric homeomorphism $\varphi : [0, 1] \to \gamma$.*

Remark 124. *Actually, from the above proof of γ is doubling we can obtain that the Assouad dimension $dim_A(\gamma) \leq dim_H(\gamma) \leq 2$.*

Now we give a construction for a Ptolemy segment in the plane with the given menger curvature.

Let a continuous function $K(t) : [0, a] \to (0, +\infty)$ be given. Suppose a curve $\gamma : [0, a] \to \mathbb{R}^2$ parameterized by distance can be formulated as

$$\gamma(t) = (t, f(t))$$

where $f(0) = a$ and $f(a) = 0$.

The distance between $\gamma(t)$ and $\gamma(s)$ is defined as

$$d(\gamma(t), \gamma(s)) = sf(t) - tf(s), t \leq s.$$

Easy to check that γ is a Ptolemy segment. Now let us find the proper $f(t)$ which satisfies our requirement.

Choose $x, y, z \in \gamma$ such that $d(x, y) = d(y, z)$. Hence we have

$$d(x, y) = tf(t - \Delta t) - (t - \Delta t)f(t), \tag{9.1}$$

$$d(y, z) = (t + \Delta s)f(t) - tf(t + \Delta s), \tag{9.2}$$

$$d(x, z) = (t + \Delta s)f(t - \Delta t) - (t - \Delta t)f(t + \Delta s) \tag{9.3}$$

Then

$$c^2(x, y, z) = \frac{d(x, y) + d(y, z) - d(x, z)}{d(x, y)^3} \tag{9.4}$$

$$= \frac{(\Delta t + \Delta s)f(t) - \Delta t f(t + \Delta s) - \Delta s f(t - \Delta t)}{(tf(t - \Delta t) - (t - \Delta t)f(t))^3} \tag{9.5}$$

from the condition $d(x, y) = d(y, z)$, we get that

$$\frac{\Delta t}{\Delta s} \to 1$$

Hence

$$\begin{aligned} K^2(t) &= \lim_{\Delta t, \Delta s \to 0} c^2(x, y, z) \\ &= \frac{f''}{(tf' + f)^3} \end{aligned}$$

According to the theory of differential equations, the last equation can be solved and hence unique. Now we finished our construction.

Chapter 10

Hyperbolic Approximation and Hyperbolic Cones

We summarize the hyperbolic approximation with the known results which has been done by [BS1] and [JJ] to compare with the second section of hyperbolic cone constructions.

10.1 Hyperbolic Approximation

The idea of a hyperbolic approximation is to construct for a given metric space Z a Gromov hyperbolic space X such that $\partial_\infty X = Z$, in some suitable sense.

This procedure appears in the literature through various sorts of cone construction on Z, which are a sort of a warped products analogue adapted to the "rough" setting of Gromov hyperbolic spaces. Classical sources for this approach include [TV1] and [BS1]. In [BS1] Buyalo and Schroeder further developed constructions of Elek [E2], and Bourdon and Pajot [BP] to give a very intuitive geometric construction of such a space X. Not only is their approach to the construction very elementary and illuminating, it also produces a particularly nice space X, namely a metric graph. Probably just because of the elementary nature of this construction, we obtain quite a clear view on the structure of X.

Moreover, since the operation of taking a boundary at infinity of a Gromov-hyperbolic space canonically gives rise to *quasi-metrics* on the boundary (rather than honest metrics) we would like to be able to perform hyperbolic approximation directly on such a quasi-metric space, instead of first introducing a non-canonical visual metric and *then* approximating this latter metric space. It is another feature of Buyalo and Schroeder's construction that it translates readily to the setting of quasimetric spaces.

10.1.1 The Construction

Let (Z, ρ) be a complete K-quasimetric space. Let $r \leq 1/K^3$. The procedure now goes as follows. For every $k \in \mathbb{Z}$ let V_k be a maximal r^k-separated subset of Z (such exist by Zorn), where r^k-separated means $\rho(v, v') \geq r^k$ for all $v, v' \in V_k$. Denote by $\mathcal{V}$ the set of all ordered pairs (k, z) with $k \in \mathbb{Z}$ and $z \in V_k$. The projection $\ell : \mathcal{V} \to \mathbb{Z}$ to the first coordinate is called *level function*, and $\ell(v)$ the *level* of v, while the projection $\pi : \mathcal{V} \to Z$ to the second coordinate sends v to its *center* $\pi(v) \in Z$.

Remark 125. *Sometimes the notation $\pi(v)$ becomes too cumbersome so that we often identify a point $v \in V_k$ with its center $\pi(v) \in Z$. The notation $\rho(v, w)$ is thus interpreted to mean $\rho(\pi(v), \pi(w))$.*

Remark 126 (Hereditary vertex systems)**.** *Also by a Zorn-type argument there exist* hereditary *vertex systems $\mathcal{V} = \{V_k\}_k$, meaning that $\pi(V_k) \subset \pi(V_{k+1})$. Working with such hereditary systems often simplifies arguments and we will use them without reservation when it suits us.*

The *hyperbolic approximation of Z with parameter $r \leq 1/K^3$*, denoted $\mathrm{H}_r(Z, \rho)$ or $\mathrm{H}_r(Z)$ for short, is now defined to be the simplicial graph with vertex set $\mathcal{V}$, where two vertices $v, w \in \mathcal{V}$ are joined by an edge exactly when

- $\ell(v) = \ell(w)$ and the sets $B(v) := B_{Kr^{l(v)}}(\pi(v))$ and $B(w) := B_{Kr^{l(w)}}(\pi(w))$ intersect in Z, or
- $\ell(v) = \ell(w) + 1$ and $B(v)$ is contained in $B(w)$.

The first point is slightly different from the definition in [BS], §6.1. We opt to use open balls for technical reasons.

10.1.2 Metric Structure of $\mathrm{H}(X)$

For Z a *metric* space, Buyalo and Schroeder proved that $\mathrm{H}_r(Z)$ is a hyperbolic space with all of the desired properties (i.e. Thm. 133 holds). Even though the proofs are easily adapted to the quasimetric setting, we here include, for the sake of completeness, the rewritten proofs of the lemmata in [BS1], §§6.2, 6.3 which lead up to the desired Theorem 133.

Lemma 127 ([BS1] Lemma 6.2.1)**.** *For every $v \in V$ there is a vertex $w \in V$ with $\ell(w) = \ell(v) - 1$ radially connected to any horizontal neighbor of v.*

Proof. Let $\ell(v) = k + 1$ and $w \in V_k$ such that $\rho(v, w) < r^k$ and v' a horizontal neighbor of v. Then $z \in B(v')$ means $\rho(z, v') < Kr^{k+1}$. Let $s \in B(v) \cap B(v')$.

$$\begin{aligned}
\rho(z, w) &\leq K \max\{\rho(w, v), \rho(v, z)\} \\
&\leq K \max\Big\{\rho(w, v), K \max\{\rho(v, v'), \rho(v', z)\}\Big\} \\
&\leq K \max\left\{\rho(w, v), K \max\Big\{K \max\{\rho(v, s), \rho(s, v')\}, \rho(v', z)\Big\}\right\},
\end{aligned}$$

which, since $\rho(v, s), \rho(s, v') < Kr^{k+1}$, implies $\rho(z, w) < Kr^k$, where we used that $K^4 r^{k+1} \leq Kr^k$, i.e. $r \leq 1/K^3$.

□

Lemma 128 ([BS1] Lemma 6.2.2)**.** *For every $v, v' \in \mathcal{V}$ there exists $w \in \mathcal{V}$ with $\ell(w) \leq \ell(v), \ell(v')$ such that v, v' can be connected to w by radial geodesics. In particular, the space X is geodesic.*

Proof. Let $\ell(v) = k$ and $\ell(v') = k'$. Choose $m < \min\{k, k'\}$ small enough such that $\rho(v, v') \leq r^{m+1}$. Applying Lemma 127, we find radial geodesics $\gamma = v_k v_{k-1} \dots v_m$ and $\gamma' = v'_{k'} v'_{k'-1} \dots v_{m'}$ in X connecting $v = v_k$ and $v' = v_{k'}$ respectively with the m-th level. It follows from the definition of radial edges that $v \in B(u)$, $v' \in B(u')$ for every vertex $u \in \gamma, u' \in \gamma'$. Then

$$\begin{aligned}\rho(v', v_m) &\leq K \max\{\rho(v_m, v_{m+1}), \rho(v_{m+1}, v')\} \\ &\leq K \max\{\rho(v_m, v_{m+1}, K \max\{\rho(v_{m+1}, v), \rho(v, v')\}\},\end{aligned}$$

hence $\rho(v_m, v') < K r^m$. □

Lemma 129 ([BS1] Lemma 6.2.3)**.** *Assume that $|vv'| \leq 1$ are horizontal neighbors. Then any w, w' radially connected to v and v' respectively are horizontal neighbors if $\ell(w) = \ell(w')$.*

Proof. $B(v) \cap B(v') \neq \emptyset$ and $B(v) \subset B(w), B(v') \subset B(w')$ imply $B(w) \cap B(w') \neq \emptyset$. □

Corollary 130 ([BS1] Corollary 6.2.4)**.** *For any two radial geodesics γ, γ' with common ends, the distance between vertices of common levels is at most 1.* □

The rest of [BS1] §6.2, namely Lemmata, Corollaries and Propositions 6.2.5-6.2.10 merely rely on the results we just proved and do not involve any details about the exact definition of the graph X, thus their proofs need not be repeated here.

In the same vein we can adapt the proofs of the Lemmata in [BS1] §6.3 to the quasimetric setting. It then follows from [BS1] Thms. 6.3.1, 6.4.1, (cf. Thm. 133 below) that $\partial_\infty^{1/r} \mathrm{H}_r(Z, \rho)$ is bilipschitz equivalent to (Z, ρ). So far this only holds for $r \leq 1/K^3$. Now the boundaries at infinity come equipped with a family of quasimetrics $a^{-(\cdot|\cdot)}$ for $a > 1$. The corresponding situation for hyperbolic approximations is that they should be taken for a family of parameters $r \in (0, 1)$, not just for $r \in (0, 1/K^3]$. Even though it should intuitively be possible to make a similar construction with balls as above, it seems the resulting graph is too difficult to control. For this reason, we resort to a scaling trick.

First of all we find it convenient to use $r = 1/K^3$ as a fixed reference for r.

Definition 131. *Let (Z, ρ) a complete K-quasi-metric space and $r \in (0, 1)$. Let $l(r, K) := \log_r(1/K^3) = -\frac{\log K^3}{\log r}$.*

The hyperbolic approximation of (Z, ρ) with parameter r, H_r^K, is defined to be the graph of the hyperbolic approximation of $(Z, \rho^{1/l})$ with parameter r as described above, but scaled so that each edge has length $l = l(r, K)$.

Remark 132. *The graph $\mathrm{H}_r(Z, \rho)$ does not depend on the choice of vertex system $\mathcal{V}$.*

Also, it follows from the bilipschitz Extension Theorems that the hyperbolic approximation is independent of the quasimetric constant K used, i.e. $\mathrm{H}_r^K(Z, \rho)$ is roughly isometric to $\mathrm{H}_r^{K'}(Z, \rho)$ if ρ is both a K- and a K'-quasimetric on Z. Furthermore, these same Extension Theorems immediately yield that approximations w.r.t. different parameters r, r' are merely scalings of each other, more precisely

$$\mathrm{H}_r(Z, \rho) = \frac{\ln r}{\ln r'} \mathrm{H}_{r'}(Z, \rho).$$

The following fundamental theorem summarizes the properties of the hyperbolic approximation.

Theorem 133 (Compare [BS1] Thms. 6.3.1, 6.4.1)**.** *Let (Z, ρ) be a complete quasimetric space, $r \in (0, 1)$. The hyperbolic approximation $\mathrm{H}_r(Z)$ is a visual geodesic hyperbolic space and there is a canonical identification $\partial_\infty \mathrm{H}_r(Z) = Z$ of sets. Moreover, if (Z, ρ) is extended then for any $b \in \mathcal{B}(\omega)$, $\partial_\infty^{1/r,b} \mathrm{H}_r(Z, \rho)$ and (Z, ρ) are bilipschitz equivalent. If (Z, ρ) is not extended, then $\partial_\infty^{1/r,o} \mathrm{H}_r(Z, \rho)$ and (Z, ρ) are bilipschitz equivalent.*

The moral of the story is that, given a complete quasimetric space (Z, ρ), there is for every $a > 1$ exactly one (up to rough isometry) visual geodesic hyperbolic space X such that $\partial_\infty^a X$ is bilipschitz-quasimoebius to (Z, ρ), and the "functor" $\mathrm{H}_{1/a}$ spits out exactly this space X when applied to (Z, ρ).

In the case of extended Z, $\mathrm{H}(Z)$ has a distinguished boundary point ω corresponding to the infinitely remote point ξ of Z, while in the non-extended case the root o of the approximation will serve as distinguished base point.

10.2 Hyperbolic cones over Möbius spaces

In this chaper we prove Theorem 3. Therefore give (bases on [BS]) a construction, how to associate to a ptolemaic Möbius space $(Z, \mathcal{M})$ a hyperbolic space X (which turns out to be asymptotically PT_{-1}), such that $(Z, \mathcal{M})$ is the canonical Möbius structure of $\partial_\infty X$.

Let $(Z, \mathcal{M})$ be a complete ptolemaic Möbius space. We choose a point $\omega \in Z$ and an extended metric $d \in \mathcal{M}$ from the Möbius structure, such that $\{\omega\} = \Omega(d)$ is the point at infinity. Such a metric exists by Theorem 58 and this metric is unique (up to homothety) by Lemma 57.

We take now the metric space (Z_ω, d), where $Z_\omega = Z \setminus \{\omega\}$ and apply the cone construction of [BS] to it. The space $\mathrm{Con}(Z_\omega)$ has properties analogous to the hyperbolic convex hull of a set in the boundary of a real hyperbolic space. Set

$$\mathrm{Con}(Z_\omega) = Z_\omega \times (0, \infty).$$

Define $\rho : \mathrm{Con}(Z_\omega) \times \mathrm{Con}(Z_\omega) \to [0.\infty)$ by

$$\rho((z, h), (z', h')) = 2 \log\left(\frac{d(z, z') + h \vee h'}{\sqrt{hh'}}\right). \tag{10.1}$$

It turns out that ρ satisfies the triangle inequality and is thus a metric, see [BS]. We write $|zz'| = d(z, z')$ for $z, z' \in Z_\omega$.

Proposition 134. $(\mathrm{Con}(Z_\omega), \rho)$ *is aymptotically* PT_{-1}.

Proof. Given arbitrary four points $x_i = (z_i, h_i) \in \mathrm{Con}((Z_\omega, d)), z_i \in (Z_\omega, d)$, $i = 1, 2, 3, 4$, we have

$$e^{\frac{\rho(x_i, x_j)}{2}} = \frac{|z_i z_j| + h_i \vee h_j}{\sqrt{h_i h_j}}, \quad i \neq j.$$

i.e.

$$|z_i z_j| = \sqrt{h_i h_j} e^{\frac{\rho(x_i, x_j)}{2}} - h_i \vee h_j, \quad i \neq j. \tag{10.2}$$

Since $(Z, \mathcal{M})$ is a complete ptolemaic Möbius space, (Z_ω, d) is a complete metric space which satisfies the PT_0 inequality, hence we obtain

$$|z_1z_2|\,|z_3z_4| \;+\; |z_1z_4|\,|z_2z_3| \;\geq\; |z_1z_3|\,|z_2z_4|.$$

Replacing $|z_iz_j|$ by (10.2), we have the following inequality

$$\begin{aligned}
&(\sqrt{h_1h_2}e^{\frac{\rho(x_1,x_2)}{2}} - h_1 \vee h_2)(\sqrt{h_3h_4}e^{\frac{\rho(x_3,x_4)}{2}} - h_3 \vee h_4) \\
&\qquad + (\sqrt{h_1h_4}e^{\frac{\rho(x_1,x_4)}{2}} - h_1 \vee h_4)(\sqrt{h_2h_3}e^{\frac{\rho(x_2,x_3)}{2}} - h_2 \vee h_3) \\
&\qquad\qquad \geq (\sqrt{h_1h_3}e^{\frac{\rho(x_1,x_3)}{2}} - h_1 \vee h_3)(\sqrt{h_2h_4}e^{\frac{\rho(x_2,x_4)}{2}} - h_2 \vee h_4).
\end{aligned}$$

This can be written as

$$\begin{aligned}
&\sqrt{h_1h_2h_3h_4}(e^{\frac{\rho(x_1,x_2)}{2}+\frac{\rho(x_3,x_4)}{2}} + e^{\frac{\rho(x_1,x_4)}{2}+\frac{\rho(x_2,x_3)}{2}} - e^{\frac{\rho(x_1,x_3)}{2}+\frac{\rho(x_2,x_4)}{2}}) \\
&- \sqrt{h_1h_2}(h_3 \vee h_4)e^{\frac{\rho(x_1,x_2)}{2}} - \sqrt{h_3h_4}(h_1 \vee h_2)e^{\frac{\rho(x_3,x_4)}{2}} - \sqrt{h_1h_4}(h_2 \vee h_3)e^{\frac{\rho(x_1,x_4)}{2}} \\
&- \sqrt{h_2h_3}(h_1 \vee h_4)e^{\frac{\rho(x_2,x_3)}{2}} + \sqrt{h_1h_3}(h_2 \vee h_4)e^{\frac{\rho(x_1,x_3)}{2}} + \sqrt{h_2h_4}(h_1 \vee h_3)e^{\frac{\rho(x_2,x_4)}{2}} \\
&\qquad + (h_1 \vee h_2)(h_3 \vee h_4) + (h_1 \vee h_4)(h_2 \vee h_3) - (h_1 \vee h_3)(h_2 \vee h_4) \geq 0.
\end{aligned}$$

Using again (10.2) we obtain

$$\begin{aligned}
&e^{\frac{\rho(x_1,x_2)}{2}+\frac{\rho(x_3,x_4)}{2}} + e^{\frac{\rho(x_1,x_4)}{2}+\frac{\rho(x_2,x_3)}{2}} - e^{\frac{\rho(x_1,x_3)}{2}+\frac{\rho(x_2,x_4)}{2}} \\
&\quad \geq \frac{(h_3 \vee h_4)|z_1z_2| + (h_1 \vee h_2)|z_3z_4| + (h_2 \vee h_3)|z_1z_4| + (h_1 \vee h_4)|z_2z_3|}{\sqrt{h_1h_2h_3h_4}} \\
&\qquad\qquad - \frac{(h_2 \vee h_4)|z_1z_3| + (h_1 \vee h_3)|z_2z_4|}{\sqrt{h_1h_2h_3h_4}} \\
&\qquad + \frac{(h_1 \vee h_2)(h_3 \vee h_4) + (h_1 \vee h_4)(h_2 \vee h_3) - (h_1 \vee h_3)(h_2 \vee h_4)}{\sqrt{h_1h_2h_3h_4}}
\end{aligned} \tag{10.3}$$

Since $(a \vee b)(c \vee d) = ac \vee ad \vee bc \vee bd, a, b, c, d \in \mathbb{R}$, we easily obtain that

$$(h_1 \vee h_2)(h_3 \vee h_4) + (h_1 \vee h_4)(h_2 \vee h_3) \geq (h_1 \vee h_3)(h_2 \vee h_4)$$

which shows that the last term in (10.3) is nonnegative an can be omitted. We use below that

$$(h_i \vee h_j) + \sqrt{h_ih_j} \geq h_i + h_j$$

for all $h_i, h_j \geq 0$.
Let $\rho = \max_{i,j} \rho_{i,j}$. Then again by (10.2) $|z_iz_j| \leq \sqrt{h_ih_j}e^{\frac{1}{2}\rho}$ and thus

$$\begin{aligned}
&e^{\frac{\rho(x_1,x_2)}{2}+\frac{\rho(x_3,x_4)}{2}} + e^{\frac{\rho(x_1,x_4)}{2}+\frac{\rho(x_2,x_3)}{2}} - e^{\frac{\rho(x_1,x_3)}{2}+\frac{\rho(x_2,x_4)}{2}} + 4e^{\frac{1}{2}\rho} \\
&\quad \geq \frac{(h_3 \vee h_4)|z_1z_2| + (h_1 \vee h_2)|z_3z_4| + (h_2 \vee h_3)|z_1z_4| + (h_1 \vee h_4)|z_2z_3|}{\sqrt{h_1h_2h_3h_4}} \\
&- \frac{(h_2 \vee h_4)|z_1z_3| + (h_1 \vee h_3)|z_2z_4|}{\sqrt{h_1h_2h_3h_4}} + \frac{|z_1z_2|}{\sqrt{h_1h_2}} + \frac{|z_3z_4|}{\sqrt{h_3h_4}} + \frac{|z_1z_4|}{\sqrt{h_1h_4}} + \frac{|z_2z_3|}{\sqrt{h_2h_3}} \\
&\quad \geq \frac{(h_3 + h_4)|z_1z_2| + (h_1 + h_2)|z_3z_4| + (h_2 + h_3)|z_1z_4| + (h_1 + h_4)|z_2z_3|}{\sqrt{h_1h_2h_3h_4}} \\
&\qquad\qquad\qquad - \frac{(h_2 \vee h_4)|z_1z_3| + (h_1 \vee h_3)|z_2z_4|}{\sqrt{h_1h_2h_3h_4}} \geq 0
\end{aligned}$$

Therefore X is asymptotic PT_{-1}. $\square$

To finish the proof of Theorem 3, we have to show that $\partial_\infty X$ can be canonically identified with Z.

We chose a basepoint $z_0 \in Z_\omega$ and then $o := (z_0, 1)$ as basepoint of X. We define for simplicity $|z| := |zz_0|$. For $x = (z, h)$ and $x' = (z', h')$ in X we compute

$$(x|x')_o \;=\; \log(\frac{(|z| + h \vee 1)(|z'| + h' \vee 1)}{|zz'| + h \vee h'}). \tag{10.4}$$

Lemma 135. *A sequence $x_i = (z_i, h_i)$ in X converges at infinity, if and only if one of the following holds*

1. (z_i) is a Cauchy sequence in Z_ω and $h_i \to 0$.

2. $(|z_i| + h_i) \;\to\; \infty$.

Proof. We show first the *if* implication:

Assume 1. that (z_i) is a cauchy sequence and $h_i \to 0$. Then equation (10.4) immediately implies that $\lim_{i,j\to\infty}(z_i|z_j)_o = \infty$.

Assume 2. that $(|z_i| + h_i) \to \infty$. For given i, j let

$$M_{i,j} = \max\{(|z_i| + h_i \vee 1), (|z_j| + h_j \vee 1)\},$$

$$m_{i,j} = \min\{(|z_i| + h_i \vee 1), (|z_j| + h_j \vee 1)\}.$$

One easily sees

$$M_{i,j} \;\geq\; \frac{1}{4}(|z_i zj| + h_i \vee h_j)$$

thus

$$(x_i|x_j)_o \;=\; \log(\frac{m_{i,j}M_{i,j}}{|z_i z_j| + h_i \vee h_j}) \;\geq\; \log(\frac{1}{4}m_{i,j})$$

and hence $\lim_{i,j\to\infty}(x_i|x_j)_o = \infty$.

For the *only if* part assume that we have given a sequence $x_i = (z_i, h_i)$ with $\lim_{i,j\to\infty}(x_i|x_j)_o = \infty$.

We first show that there cannot exist two subsqeunces (x_{i_k}) and (x_{i_l}) of (x_i), such that $|z_{i_k}| + h_{i_k} \to \infty$ for $k \to \infty$ and $|z_{i_l}| + h_{i_l} \leq M$ for all l. If to the contrary such sequences would exists, then we easily obtain using triangle inequalities that

$$|z_{i_k}| + h_{i_k} \vee 1 - 2M - 1 \leq |z_{i_k} z_{i_l}| + h_{i_k} \vee h_{i_l} \leq |z_{i_k}| + h_{i_k} \vee 1 + 2M + 1$$

and hence $\limsup(x_{i_k}|x_{i_l})_o$ is finite, a contradiction.

Thus either $|z_i| + h_i \to \infty$ and we are in case 2 or $|z_i| + h_i$ is bounded. The boundedness and $(x_i|x_j)_o \to, \infty$ implies $(|z_i z_j| + h_i \vee h_j) \to \infty$ and hence (z_i) is a Cauchy sequence and $h_i \to 0$. $\square$

Lemma 136. *One can identify Z with $\partial_\infty X$ in a canonical way.*

Proof. We define a map $\chi : Z \to \partial_\infty X$ by $z \mapsto [(z, \frac{1}{i})]$ for $z \in Z_\omega$ and $\omega \mapsto [(z_0, i)]$; here [ ] denotes the equivalence class of the corresponding sequences. Formula (10.4) shows that this map is injective. Let now $\xi \in \partial_\infty X$ be given and be represented by a sequence $x_i = (z_i, h_i)$. If $|z_i| + h_i \to \infty$ then $(x_i|(z_0, i))_o \to \infty$ and $\xi = \chi(\omega)$. If $h_i \to 0$ and (z_i) a Cauchy sequence in Z_ω, then the $z := \lim z_i$ exists, since $(Z, \mathcal{M})$ is a complete Möbius structure. One easily checks $\xi = \chi(z)$. $\square$

Lemma 137. *The canonical Möbius structure of $\partial_\infty X$ equals to $\mathcal{M}$.*

Proof. We consider on $\partial_\infty X$ the canonical Möbius structure which is given by the metric $\rho_o(z,z') = e^{-(z|z')_o}$. Using metric involution we consider the extended metric in the same Möbius class with ω as infinitely remote point. This metric is given for $z, z' \in Z_\omega$ by

$$\rho_{\omega,o}(z,z') \;=\; \frac{\rho_o(z,z')}{\rho_o(\omega,z)\rho_o(\omega,z')} \;=\; e^{-(z|z')_{\omega,o}}.$$

Now

$$(z|z')_{\omega,o} = (z|z')_o \,-\, (\omega|z)_o \,-\, (\omega|z')_o.$$

By formula (10.4) we have

$$(\omega|z)_o = \lim_{i\to\infty} \log(\frac{i(|z_i|+1)}{|z_i|+i}) \;=\; \log(|z|+1)$$

and in the same way $(\omega|z')_o = \log(|z'|+1)$. Using formula (10.4) we see that for $z, z' \in Z_\omega$

$$(z|z')_o \;=\; \log(\frac{(|z|+1)(|z'|+1)}{|zz'|}.$$

Now we easily compute

$$(z|z')_{\omega,o} \;=\; -\log(|zz'|),$$

and hence

$$\rho_{\omega,o}(z,z') \;=\; |zz'|.$$

□

Bibliography

[BCL] Keith Ball, Eric A. Carlen and Elliott H. Lieb *Sharp uniform convexity and smoothness inequalities for trace norms*, Invent math. 115, 463–482 (1994).

[BF] M. Bonk and T. Foertsch *Asymptotic upper curvature bounds in coarse geometry*, Math. Z. 2006(253), 753-785.

[BFW] S. M. Buckley, K. Falk, D. J. Wraith, *Ptolemaic spaces and* CAT(0), Glasgow J. Math. 51 (2009), 301–314.

[BH] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature* Springer(1999).

[BN] I. D. Berg, I. G. Nikolaev, *Quasilinearization and curvature of Aleksandrov spaces*, Geom. Dedicata 133 (2008), 195–218.

[BP] M. Bourdon and H. Pajot *Cohomologie l_p et espaces de Besov*, J. reine angew. Math. 2003(558) 85-108.

[BS] M. Bonk and O. Schramm, *Embeddings of Gromov hyperbolic spaces*, Geom. Funct. Anal. 2000(10) 266-306.

[BS1] S. Buyalo, V. Schroeder, *Elements of asymptotic geometry*, EMS Monographs in Mathematics, 2007.

[BS2] S. Buyalo, V. Schroeder, *Möbius structures and Ptolemy spaces: boundary at infinity of complex hyperbolic spaces, arXiv:1012.1699 , 2010*

[CK] Cheeger, J., Kleiner, B., *Metric differentistion, monotonicity and maps to* L^1. Invent math(2010)182:335-370.

[DM11] D. Meyer, *Bounded turning circles are weak-quasicircles*, Proceeding of the American Mathematical Society. Vol 139, No. 5 (2011), 1751-1761.

[DKS] S. Dhompongsa, W.A. Kirk and B. Sims, *Fixed points of uniformly lipschitzian mappings*, Nonlinear Analysis 65 (2006) 762-772.

[E1] P. Enflo, *On the nonexistence of uniform homeomorphisms between L_p-spaces*, Ark. Mat. 8 (1969) 103–105.

[E2] G. Elek *The l_p-cohomology and the conformal dimension of hyperbolic cones*, Geom. Dedicata. 1997(68) 263-279.

[EF] R. Espinola and A. Fernandez-Leon, CAT(κ) *spaces, weak convergence and fixed points*, Journal of Mathematical Analysis and Applications, vol. 353, no. 1, pp. 410–427, 2009.

[EN] R. Espinola and A. Nicolae *Uniform convexity of geodesic Ptolemy spaces*, To appear in J. Convex Anal.

[F] A. H. Frink, *Distance functions and the metrization problem*, Bull. Amer. Math. Soc. 1937(43) 133-142.

[FLS] Th. Foertsch, A. Lytchak and V. Schroeder, *Nonpositive curvature and the Ptolemy inequality*, Int. Math. Res. Not. IMRN 2007, no. 22, 15 pp.

[FS1] Th. Foertsch and V. Schroeder, *Hyperbolicity,* CAT(-1)*-spaces and the Ptolemy Inequality*, Math. Ann. 350 (2011), no. 2, 339–356.

[FS2] Th. Foertsch and V. Schroeder, *Group actions on geodesic Ptolemy spaces*, Trans. Amer. Math. Soc. 363 (2011), no. 6, 2891–2906.

[FS3] T. Foertsch, V. Schroeder. *Ptolemy circles and Ptolemy segments*, Archiv der Mathematik. Volume 98, Number 6 (2012), 571-581.

[G1] M. Gromov *Hyperbolic groups: Essays in group theory*, Math. Sci. Res. Inst. Publ. 8. Springer Verlag, 1987.

[G2] M. Gromov *Asymptotic invariants of infinite groups: Geometric group theory, Vol. 2 (Sussex, 1991)*, London Math. Soc. Lecture Note Ser. Cambridge Univ. Press. 1993(182) 1–295.

[GK] K. Goebel, W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, Cambridge, 1990.

[H] J. Heinonen, *Lectures on analysis on metric spaces*, Springer-Verlag, 2001.

[HL] P. Hitzelberger and A. Lytchak, *Spaces with many affine functions*, Proc. Amer. Math. Soc. 135, 2007, 2263-2271.

[JJ] Julian Jordi. Interplay between interior and boundary geometry in Gromov hyperbolic spaces. GEOMETRIAE DEDICATA Volume 149, Number 1, 129-154, 2010.

[K] K. Kuwae, *Jensens inequality on convex spaces*, preprint (2010).

[KP] W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal.68 (12) (2008), 3689–3696.

[L1] E. A. Lifshic, *A fixed point theorem for operators in strongly convex spaces.* Voronez. Gos. Univ. Trudy Mat. Fak. 16(1975), 23–28.

[L2] T. C. Lim, *Fixed point theorems for uniformly Lipschitzian mappings in* L^p *spaces.* Nolinear Analysis. Theory, Methods& Application. Vol. 7, No. 5, pp. 555–563, 1983.

[LS] Urs Lang and Viktor Schroeder, *Jung's Theorem for Alexandrov Spaces of Curvature Bounded Above*' Annals of Global Analysis and Geometry 15: 263–275, 1997.

[LX] T. C. Lim and H. K. Xu, *Uniformly lipschitzian mappings in metric spaces with uniform normal structure*, Nonlinear Anal. 25(1995), 1231–1235.

[M] Meyer, J. *Uniformly perfect boundaries of Gromov hyperbolic spaces* Ph.D Thesis, University of Zurich 2009.

[M1] Morse, H.M. *Recurrent geodesics on a surface of negative curvature*, Trans. Amer. Math. Soc. 1921(22) 84-100.

[M2] Morse, H.M. *A fundamental class of geodesics on any closed surface of genus greater than one*, Trans. Amer. Math. Soc. 1924(26) 25-60.

[M] R. Miao. *Jung's Theorem and fixed points for p-uniformly convex spaces*, arXiv:1305.1453.

[MS1] R. Miao and V. Schroeder *A flat strip theorem for Ptolemaic spaces*, To appear in Math. Z.

[MS2] R. Miao and V. Schroeder *Hyperbolic spaces and Ptolemy Möbius Structures* arXiv:1210.5133, 2012

[Oh] Shin-ichi, Ohta, *Convexities of metric spaces*, Geom Dedicata (2007) 125, 225–250.

[P1] C. M. Petty. *Characterization of geodesic arcs*, Geometriae Dedicata. Volume 5, Number 3 (1976), 347-359,

[P2] F. Paulin. *Un groupe hyperbolique est déterminé par son bord*, J. London Math. Soc. 1996(54), 50-74.

[S] I. J. Schoenberg. *On Metric Arcs of Vanishing Menger Curvature*, The Annals of Mathematics, Second Series, Vol. 41, No. 4 (Oct., 1940), pp. 715-726.

[Sa] T. Sato, *An alternative proof of Berg and Nikolaev's characterization of* CAT(0)*-spaces via quadrilateral inequality.* Arch. Math. 93 (2009), no. 5, 487–490.

[Sch] I. J. Schoenberg, *A remark on M. M. Day's characterization of inner-product spaces and a conjecture of L. M. Blumenthal*, Proc. Amer. Math. Soc. 3 (1952), 961–964.

[TV1] D. A. Trotsenko and J. Väisälä. *Upper sets and quasisymmetric maps*, Ann. Acad. Sci. Fenn. Fenn. 1999(24) 465-488.

[TV2] P. Tukia and J. V aisala *Quasisymmetric embeddings of metric spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. 5 (1980), no. 5, 97114.

[V] J. Väisälä *Quasimöbius Maps*, J. Analyse Math. 1984/85(44) 218-234.